1. **Solution**:
Let's denote the altitude by y. Due to Newton's laws we can see that internal forces, such as explosion, do not affect on the center of mass (C.M.) motion. So the C.M. moves only in the vertical direction. Using Newton's second law we get

$$\frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}M\dot{y} = M\ddot{y} = F^{(\mathrm{external})} = -Mg. \tag{1}$$

This equation of motion is a differential equation with initial values for C.M:

$$y(0) = 2000\,\mathrm{m} \qquad \text{and} \qquad \dot{y}(0) = -60\,\frac{\mathrm{m}}{\mathrm{s}}. \tag{2}$$

The general solution is obtained by integrating the equation of motion twice:

$$\begin{aligned} \ddot{y} &= -g \\ &\Leftrightarrow \\ \dot{y}(t) &= \dot{y}(0) - \int_0^t g\mathrm{d}t' \\ &= \dot{y}(0) - gt \\ &\Leftrightarrow \\ y(t) &= y(0) - \int_0^t \dot{y}(t')\mathrm{d}t' \\ &= y(0) - \int_0^t \dot{y}(0) - gt\mathrm{d}t' \\ &= y(0) + \dot{y}(0)t - \frac{1}{2}gt^2. \end{aligned} \tag{3}$$

Thus the solution to our problem is

$$\begin{aligned} y(10\,\mathrm{s}) &= 2000\,\mathrm{m} - 60\,\frac{\mathrm{m}}{\mathrm{s}} * 10\,\mathrm{s} - \frac{1}{2} * 9,81\,\frac{\mathrm{m}}{\mathrm{s}^2} * (10\,\mathrm{s})^2 \\ &\approx 909.5\,\mathrm{m}. \end{aligned} \tag{4}$$

2. **Solution**:
a) It is easy to see that

$$\mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i$$

b) Now

$$\mathbf{v}_i = \frac{\mathrm{d}\mathbf{r}_i}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} + \frac{\mathrm{d}\mathbf{r}'_i}{\mathrm{d}t} = \mathbf{V} + \mathbf{v}'_i.$$

c) First we should notice that

$$\begin{aligned}
\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} &= \frac{1}{M}\sum_i m_i(\mathbf{R} + \mathbf{r}'_i) = \mathbf{R} + \frac{1}{M}\sum_i m_i \mathbf{r}'_i \\
\Rightarrow \sum_i m_i \mathbf{r}'_i &= 0 \\
\Rightarrow \sum_i m_i \mathbf{v}'_i &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_i m_i \mathbf{r}'_i = 0.
\end{aligned} \tag{5}$$

d) Now we can easily show that

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times \mathbf{p}_i \\
&= \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i \\
&= \sum_i m_i (\mathbf{R} + \mathbf{r}'_i) \times (\mathbf{V} + \mathbf{v}'_i) \\
&= \sum_i m_i [\mathbf{R} \times \mathbf{V} + \mathbf{R} \times \mathbf{v}'_i + \mathbf{r}'_i \times \mathbf{V} + \mathbf{r}'_i \times \mathbf{v}'_i] \\
&= \mathbf{R} \times \mathbf{V} \underbrace{\sum_i m_i}_{=M} + \mathbf{R} \times \underbrace{\left(\sum_i m_i \mathbf{v}'_i\right)}_{=0} + \underbrace{\left(\sum_i m_i \mathbf{r}'_i\right)}_{=0} \times \mathbf{V} + \sum_i \mathbf{r}'_i \times \underbrace{(m_i \mathbf{v}'_i)}_{=\mathbf{p}'_i} \\
&= \mathbf{R} \times M\mathbf{V} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i.
\end{aligned} \tag{6}$$

3. **Solution**: Like before

$$\begin{aligned}
T &= \frac{1}{2}\sum_i m_i v_i^2 \\
&= \frac{1}{2}\sum_i m_i (\mathbf{V} + \mathbf{v}'_i) \cdot (\mathbf{V} + \mathbf{v}'_i) \\
&= \frac{1}{2}\sum_i m_i (V^2 + 2\mathbf{V} \cdot \mathbf{v}'_i + v'^2) \\
&= \frac{1}{2}V^2 \underbrace{\sum_i m_i}_{=M} + \mathbf{V} \cdot \underbrace{\sum_i m_i \mathbf{v}'_i}_{=0} + \frac{1}{2}\sum_i m_i v_i'^2 \\
&= \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i v_i'^2.
\end{aligned} \tag{7}$$

4. **Solution**:

a) The force is

$$\mathbf{F} = \underbrace{(6abz^3y - 20bx^3y^2)}_{=F_x}\mathbf{i} + \underbrace{(6abxz^3 - 10bx^4y)}_{=F_y}\mathbf{j} + \underbrace{(18abxz^2y)}_{=F_z}\mathbf{k}.$$

So the components of the curl are

$$\begin{aligned}
(\nabla\times\mathbf{F})_x &= \partial_y F_z - \partial_z F_y = 18abxz^3 - 18abxz^3 = 0\\
(\nabla\times\mathbf{F})_y &= \partial_z F_x - \partial_x F_z = 18abyz^3 - 18abyz^3 = 0\\
(\nabla\times\mathbf{F})_z &= \partial_x F_y - \partial_y F_x = 6abz^3 - 40bx^3y - (6abz^3 - 40bx^3y) = 0.
\end{aligned}$$

This means $\nabla\times\mathbf{F} = 0$ and thus $\mathbf{F}(x,y,z)$ is conserved. Let's find a potential function $\varphi(x,y,z)$ such that $\mathbf{F} = -\nabla\varphi = -\partial_x\varphi\mathbf{i} - \partial_y\varphi\mathbf{j} - \partial_z\varphi\mathbf{k}$. This means

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{i} + F_z\mathbf{j} = -\partial_x\varphi\mathbf{i} - \partial_y\varphi\mathbf{j} - \partial_z\varphi\mathbf{k}$$

which gives us a group of equations

$$\begin{aligned}
F_x &= -\frac{\partial\varphi}{\partial x}\\
F_y &= -\frac{\partial\varphi}{\partial y}\\
F_y &= -\frac{\partial\varphi}{\partial z}.
\end{aligned}$$

We can solve the potential function φ from each component:

$$\varphi = \begin{cases}
-\int F_x \mathrm{d}x = -6abxz^3y + 5bx^4y^2 + f(y,z)\\
-\int F_y \mathrm{d}y = -6abxz^3y + 5bx^4y^2 + g(x,z)\\
-\int F_z \mathrm{d}z = -6abxz^3y + h(x,y)
\end{cases}$$

but the force has to be the same derived from each expression meaning $f(y,z) = g(x,z) = C(\text{ constant})$ and $h(x,y) = 5bx^4y^2 + C$. Thus the solution is $\varphi(x,y,z) = -6abxz^3y + 5bx^4y^2 + C$.

b)

The force is $\mathbf{F} = -\nabla V$. Because the force F is derived from the potential function V, it is conserved by the definition (remember $\nabla\times\nabla V = 0$ for any arbitrary scalar function V). But for fun let's check that $\nabla\times\mathbf{F} = 0$. There are two ways to do this. In spherical coordinates

$$\begin{aligned}
\mathbf{F}(r,\theta,\phi) &= -\nabla V(r)\\
&= -\Big(\frac{\partial V(r)}{\partial r}\hat{r} + \frac{1}{r}\underbrace{\frac{\partial V(r)}{\partial\theta}}_{=0}\hat{\theta} + \frac{1}{r\sin\theta}\underbrace{\frac{\partial V(r)}{\partial\phi}}_{=0}\hat{\phi}\Big)\\
&= \gamma m\frac{\partial}{\partial r}\frac{1}{r}\hat{r}\\
&= -\gamma\frac{m}{r^2}\hat{r}.
\end{aligned}$$

This result means that the force has spherical symmetry i.e. the force has only a radial component and $\mathbf{F}(|\mathbf{r}|)$. It is easy to check that

$$\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}(\frac{\partial}{\partial\theta}(F_\phi \sin\theta) - \frac{\partial F_\theta}{\partial\phi})\hat{r} + \frac{1}{r}(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{\partial}{\partial r}(rF_\phi))\hat{\theta} + \frac{1}{r}(\frac{\partial}{\partial r}(rF_\phi)) - \frac{\partial F_r}{\partial\theta})\hat{\theta} = 0$$

when we remember $F_\theta = F_\phi = 0$ (the result above) and $\partial_\theta F_r = \partial_\phi F_r = 0$.
In Cartesian coordinates it is a little bit more difficult:

$$F_x = -\frac{\partial}{\partial x}(-\gamma\frac{m}{r}) = \gamma m\frac{\partial}{\partial x}(x^2+y^2+z^2)^{-\frac{1}{2}} = \gamma m(-\frac{1}{2})2x(x^2+y^2+z^2)^{-\frac{3}{2}} = -\gamma m\frac{x}{r^3}.$$

In a similar manner one finds out that

$$F_y = -\gamma m\frac{y}{r^3} \qquad \text{and} \qquad F_z = -\gamma m\frac{z}{r^3}.$$

Thus

$$\mathbf{F} = -\nabla V = -\gamma m\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3} = -\gamma m\frac{\mathbf{r}}{r^3} = -\gamma\frac{m}{r^2}\hat{r}$$

that is the same result as in spherical coordinates (as it should be). Now

$$\partial_y F_z = -\gamma mz\partial_y(x^2+y^2+z^2)^{-\frac{3}{2}} = -\gamma mz(-\frac{3}{2})2y(x^2+y^2+z^2)^{-\frac{5}{2}} = 3\gamma m\frac{yz}{r^5}$$

and

$$\partial_z F_y = -\gamma my\partial_z(x^2+y^2+z^2)^{-\frac{3}{2}} = -\gamma my(-\frac{3}{2})2z(x^2+y^2+z^2)^{-\frac{5}{2}} = 3\gamma m\frac{yz}{r^5}$$

$$\Rightarrow (\nabla \times \mathbf{F})_x = \partial_y F_z - \partial_z F_y = 0$$

and similarly the y- and z-component of the curl are zero.

c)

In mechanics the work W_{12} done by the field $\mathbf{F}$ on the particle between times $t_1 = 1$ and $t_2 = 2$ is the line integral

$$\begin{aligned} W_{12} &= \int_{r_1}^{r_2} \mathbf{F}\cdot \mathrm{d}\mathbf{r} \\ &= \int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{v}\mathrm{d}t \\ &= \int_1^2 [F_x v_x + F_y v_y + F_z v_z]\mathrm{d}t \\ &= \int_1^2 [96t^5 + (324t^3 - 288t^2 - 144t^2 + 128t) + 36t^3]\mathrm{d}t \\ &= \int_1^2 [96t^5 + 360t^3 - 432t^2 + 128t]\mathrm{d}t \\ &= \cdots = 1542. \end{aligned} \tag{8}$$

On the other hand, if we notice $\mathbf{F} = 2\dot{\mathbf{v}}$ we can think that $\mathbf{F}$ is a force acting on a particle whose "mass" is two. The work is

$$W_{12} = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \mathrm{d}t = \int_{t_1}^{t_2} 2\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \cdot \mathbf{v} \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v} \cdot \mathbf{v}) \mathrm{d}t = \Big/_{1}^{2} v^2 = \cdots = 1542 \tag{9}$$

5. **Solution**:

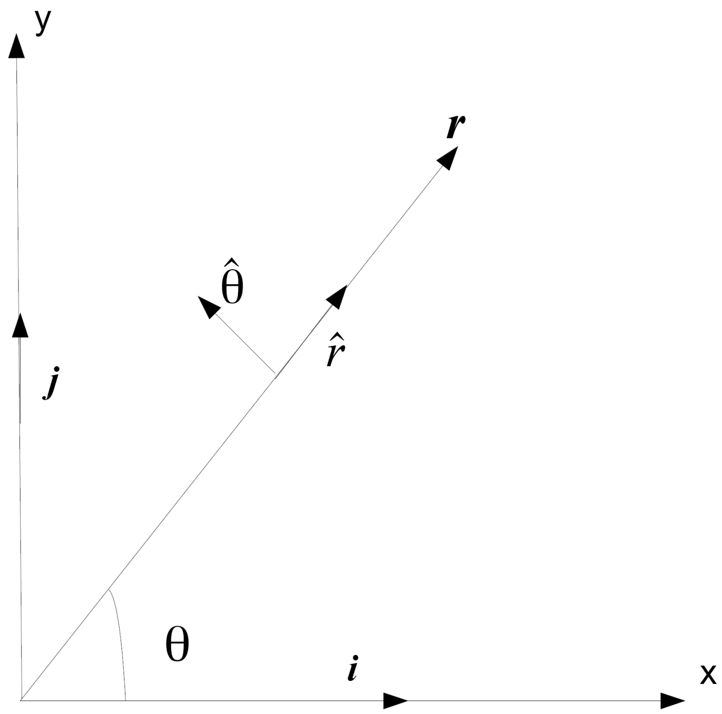

Figure 1: *Cartesian versus polar coordinate system.*

We can represent $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$ and from the figure we can see that $x = r\cos\theta$ and $y = r\sin\theta$. So $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = r\hat{r}$. Now $|\hat{r}| = \sqrt{\hat{r} \cdot \hat{r}} = \sqrt{\cos^2\theta + \sin^2\theta} = 1$ as we want it to be . We want that $\hat{\theta}$ is in the direction of increasing angle θ, so we need to calculate derivative of $\hat{r}$ with respect to θ:

$$\hat{\theta} = \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}. \tag{10}$$

Now

$$\begin{aligned} 2\hat{r} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} &= \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} \cdot \hat{r} + \hat{r} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\hat{r} \cdot \hat{r}) = \frac{\mathrm{d}}{\mathrm{d}\theta} \underbrace{|\hat{r}|}_{=1} = 0 \\ \Rightarrow \hat{r} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} &= 0 \end{aligned} \tag{11}$$

or straightforwardly calculating

$$\begin{aligned} \hat{r} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} &= (\cos\theta \mathbf{i} + \sin\theta \mathbf{j}) \cdot (-\sin\theta \mathbf{i} + \cos\theta \mathbf{j}) \\ &= -\sin\theta\cos\theta + \sin\theta\cos\theta \\ &= 0 \end{aligned} \tag{12}$$

and

$$\left|\frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta}\right| = \sqrt{\frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta} \cdot \frac{\mathrm{d}\hat{r}}{\mathrm{d}\theta}} = \sqrt{\sin^2\theta + \cos^2\theta} = 1. \tag{13}$$

Thus it is a good choice to define $\hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ ($\hat{\theta} \perp \hat{r}$ and $|\hat{\theta}| = 1$), and so $\hat{r}$ and $\hat{\theta}$ form an *orthonormal basis*.

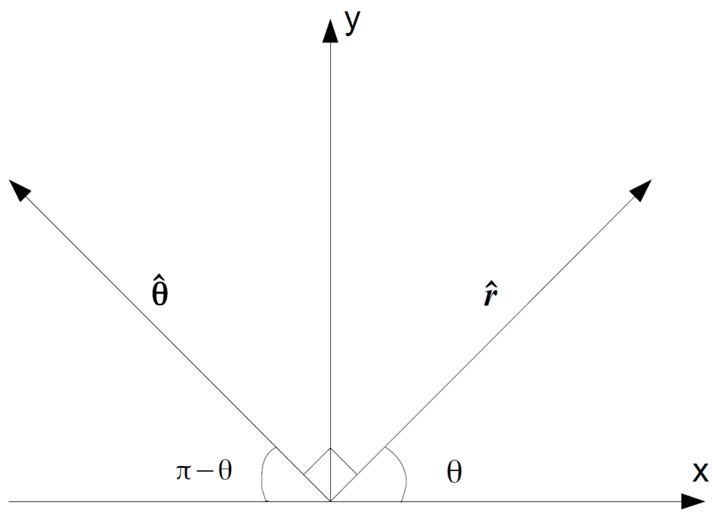

Figure 2: *The unit vectors in polar coodinates.*

Using the knowledge on Figure 2 we can also deduce by geometry that $\hat{\theta} = \hat{r}(\pi - \theta)$ and define our second vector simply just using the formula of the radial vector $\hat{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ by making the replacement $\theta \to \pi - \theta$. This will give the same result as above. The vector $\hat{\theta}$ will be automatically a unit vector because the length of the radial vector $\hat{r}$ is one. Defining the vector $\hat{\theta}$ this way we make also sure that $\hat{\theta} \perp \hat{r}$.
Then to solve $\mathbf{i}$ and $\mathbf{j}$ in terms of $\hat{r}$ and $\hat{\theta}$ we have to solve the group of equations:

$$\begin{cases} \hat{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \hat{\theta} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \end{cases} \tag{14}$$

This is easy to solve as a pair of linear equations for the two unknown variables $\mathbf{i}$ and $\mathbf{j}$. It can be done with matrix algebra:

$$\begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ -\sin\theta\mathbf{i} + \cos\theta\mathbf{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = A \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix}, \tag{15}$$

where A is an orthogonal matrix $A^{-1} = A^{\mathrm{T}}$. So

$$\begin{pmatrix} \mathbf{i} \\ \mathbf{j} \end{pmatrix} = A^{-1} \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = A^{\mathrm{T}} \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \hat{r} \\ \hat{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta\hat{r} - \sin\theta\hat{r} \\ \sin\theta\hat{r} + \cos\theta\hat{\theta} \end{pmatrix}$$

$$\Rightarrow \begin{cases} \mathbf{i} = \cos\theta\hat{r} - \sin\theta\hat{r} \\ \mathbf{j} = \sin\theta\hat{r} + \cos\theta\hat{\theta}. \end{cases} \tag{16}$$

For the time derivatives remember that θ and r both are functions of time but $\mathbf{i}$ and $\mathbf{j}$ are constant. Now the calculations:

$$\dot{\hat{r}} = \frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = -\dot{\theta}\sin\theta + \dot{\theta}\cos\theta\mathbf{j} = \dot{\theta}\hat{\theta} \tag{17}$$

$$\dot{\hat{\theta}} = \frac{\mathrm{d}}{\mathrm{d}t}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = -\dot{\theta}\cos\theta - \dot{\theta}\sin\theta\mathbf{j} = -\dot{\theta}\hat{r} \tag{18}$$

as well as the velocity

$$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(r\hat{r}) = \dot{r}\hat{r} + r\dot{\hat{r}} =^{(17)} \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \tag{19}$$

and the acceleration

$$\begin{aligned} \mathbf{a} &= \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) \\ &= \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\dot{\hat{\theta}} \\ &=^{(18)} \ddot{r}\hat{r} + 2\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r} \\ &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}. \end{aligned} \tag{20}$$