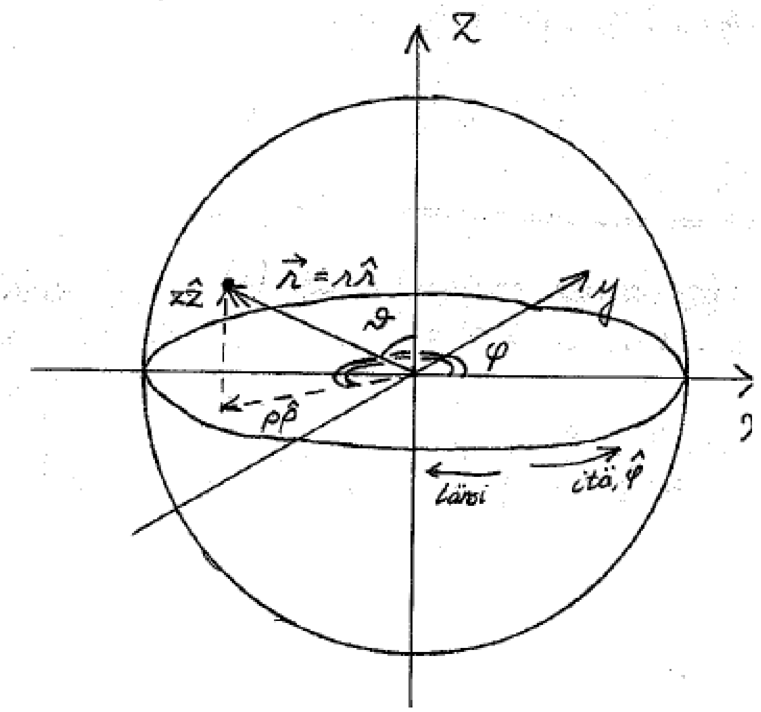
1. **Solution**:




**a)**

Let's consider an object on the Earth's surface in the location $\mathbf{r}_{\text{obj}}$. The object is on the rest meaning $\dot{\mathbf{r}}_{\text{obj}} = 0$ and it is affected by the force $m\mathbf{g}$ i.e.

$$m\ddot{\mathbf{r}}_{\text{obj}} = m\mathbf{g}.$$

The formulae in the above include also *fictitious* forces due to rotating coordinate system (the Earth) but of course also the normal external forces $\mathbf{F}^{(e)}$. In our system the only external force is gravitation:

$$\mathbf{F}^{(e)} = m\mathbf{g}_0 = -mg_0\hat{\mathbf{r}}.$$

The Earth is rotating with a constant angular velocity

$$\boldsymbol{\omega} = \omega\mathbf{k} \Rightarrow \dot{\boldsymbol{\omega}} = 0.$$

Now from the lectures we get

$$\underbrace{m\ddot{\mathbf{r}}_{\text{obj}}}_{=m\mathbf{g}} = \underbrace{\mathbf{F}^{(e)}}_{=-mg_0\hat{\mathbf{r}}} - 2m\boldsymbol{\omega} \times \underbrace{\dot{\mathbf{r}}_{\text{obj}}}_{=0} - m\underbrace{\dot{\boldsymbol{\omega}}}_{=0} \times \mathbf{r} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\Leftrightarrow$$

$$\mathbf{g} = -g_0\hat{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

For the cross product we use cylinder coordinates

$$\mathbf{r} = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}} = r\sin\theta\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$$

and thus

$$\begin{aligned}\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) &\\ &= \boldsymbol{\omega}\times[\omega\hat{\mathbf{z}}\times(r\sin\theta\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}})]\\ &= \omega\hat{\mathbf{z}}\times[\omega r\sin\theta\underbrace{(\hat{\mathbf{z}}\times\hat{\boldsymbol{\rho}})}_{=\hat{\boldsymbol{\varphi}}} + \omega z\underbrace{(\hat{\mathbf{z}}\times\hat{\mathbf{z}})}_{=0}]\\ &= \omega^2 r\sin\theta\underbrace{(\hat{\mathbf{z}}\times\hat{\boldsymbol{\varphi}})}_{=-\hat{\boldsymbol{\rho}}}\\ &= -\omega^2 r\sin\theta\hat{\boldsymbol{\rho}}.\end{aligned}$$

So it is that

$$\mathbf{g} = -g_0\hat{\mathbf{r}} + r\omega^2\sin\theta\hat{\boldsymbol{\rho}}.$$

**b)**

Let's make the perturbation

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_1 + \cdots$$

where $\mathbf{r}_i \propto \omega^i$. We assumed a coordinate system where $\hat{\mathbf{z}}$ is pointing to the direction of $-\mathbf{g}$:

$$\hat{\mathbf{z}} = -\mathbf{g}/g_0 = \hat{\mathbf{r}} - \omega^2\sin\theta g_0^{-1}\hat{\boldsymbol{\rho}} = \hat{\mathbf{r}} - O(\omega^2)\hat{\boldsymbol{\rho}} \approx \hat{\mathbf{r}},$$

because we neglect second order terms in the first order perturbation. We start again from the formula

$$m\ddot{\mathbf{r}} = -mg\hat{\mathbf{r}} - 2m\boldsymbol{\omega}\times\dot{\mathbf{r}} - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

where we used $\dot{\boldsymbol{\omega}} = 0$. In the first order perturbation we get

$$\underbrace{m\ddot{\mathbf{r}}_0}_{\propto\omega^0} + \underbrace{m\ddot{\mathbf{r}}_1}_{\propto\omega^1} = -\underbrace{mg\hat{\mathbf{r}}}_{\propto\omega^0} - 2m\underbrace{\boldsymbol{\omega}\times\dot{\mathbf{r}}_0}_{\propto\omega^1} - 2m\underbrace{\boldsymbol{\omega}\times\dot{\mathbf{r}}_1}_{\propto\omega^2} - m\underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_0)}_{\propto\omega^2} - \underbrace{\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}_1)}_{\propto\omega^3}.$$

The zeroth order equation is

$$\ddot{\mathbf{r}}_0 = -g\hat{\mathbf{r}} = -g\hat{\mathbf{z}} \to \mathbf{r}_0 = \left(h - \frac{1}{2}gt^2\right)\hat{\mathbf{z}}$$

where our initial conditions are $\dot{\mathbf{r}}_0(0) = 0$ and $\mathbf{r}_0(0) = h$ i.e. the object start to fall from the height $h$. The second order equation is

$$\begin{aligned}\ddot{\mathbf{r}}_1 &= -2\boldsymbol{\omega}\times\dot{\mathbf{r}}_0\\ &= -2\omega(\cos\theta\hat{\mathbf{z}} - \sin\theta\hat{\mathbf{x}})\times(-gt\hat{\mathbf{z}})\\ &= -2\omega gt\sin\theta\underbrace{(\hat{\mathbf{x}}\times\hat{\mathbf{z}})}_{=-\hat{\mathbf{y}}}\\ &= 2\omega gt\sin\theta\hat{\mathbf{y}}\end{aligned}$$

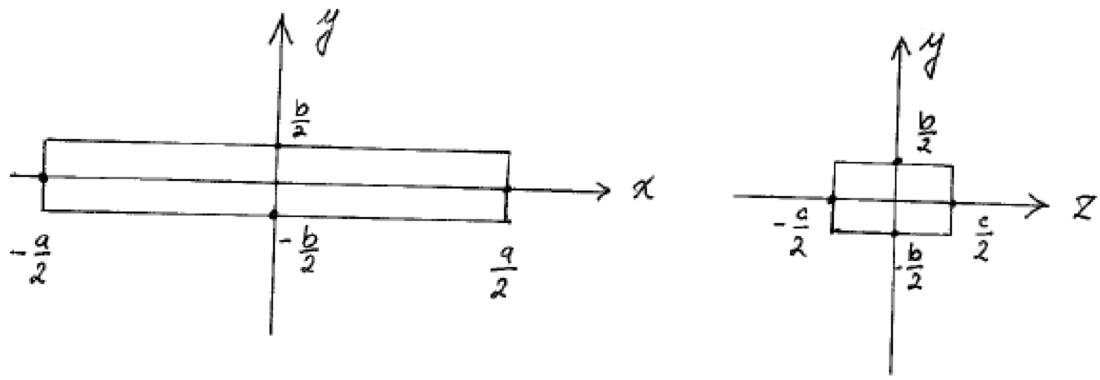
and the solution is

$$\mathbf{r}_1 = \frac{1}{3}\omega g t^3 \sin\theta \hat{\mathbf{y}}.$$

So our second-order perturbation gives the result

$$\mathbf{r} \approx \mathbf{r}_0 + \mathbf{r}_1 = \left(h - \frac{1}{2}gt^2\right)\hat{\mathbf{z}} + \frac{1}{3}\omega g t^3 \sin\theta \hat{\mathbf{y}}.$$

2. **Solution**:




The density function is

$$\rho(\mathbf{r}) = \begin{cases} M/abc, & |x| < \frac{a}{2} \cup |y| < \frac{b}{2} \cup |z| < \frac{c}{2} \\ 0, & \text{otherwise} \end{cases}$$

Let's denote $\rho_0 = M/abc = \text{constant}$ and $x_1 = x$, $x_2 = y$ and $x_3 = z$. The definition of the moment of inertia is

$$I_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \rho(\mathbf{r})(\delta_{ij}r^2 - x_i x_j)\mathrm{d}^3 r.$$

First we study the diagonal elements of the inertia tensor ($i = j$) and then we have

$$\delta_{ij} = \delta_{ii} = 1$$

and

$$r^2 = x^2 + y^2 + z^2.$$

Now we get from the definition that

$$\begin{aligned}
I_{11} &= \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2} \rho_0(x^2+y^2+z^2-x^2)\mathrm{d}z\mathrm{d}y\mathrm{d}x \\
&= \rho_0 \underbrace{\int_{-a/2}^{a/2}\mathrm{d}x}_{=a} \int_{-b/2}^{b/2}\int_{-c/2}^{c/2}(y^2+z^2)\mathrm{d}z\mathrm{d}y \\
&= \rho_0 a \int_{-b/2}^{b/2} \Big/_{-c/2}^{c/2} \left[y^2 z + \frac{1}{3}z^3\right]\mathrm{d}y \\
&= \rho_0 a \int_{-b/2}^{b/2} \left[cy^2 + \frac{1}{12}c^3\right]\mathrm{d}y \\
&= \rho_0 ac \Big/_{-b/2}^{b/2} \left[\frac{1}{3}y^3 + \frac{1}{12}c^2 y\right] \\
&= \rho_0 ac\left[\frac{1}{12}b^3 + \frac{1}{12}c^2 b\right] \\
&= \rho_0 acb\left(\frac{b^2}{12} + \frac{c^2}{12}\right) \\
&= \frac{M}{12}(b^2+c^2)
\end{aligned}$$

because $\rho_0 = M/abc \Rightarrow M = \rho_0 abc$. In a similar way we get

$$I_{22} = \frac{M}{12}(a^2+c^2) \quad \text{and} \quad I_{33} = \frac{M}{12}(a^2+b^2).$$

The off-diagonal elements $i \neq j \Rightarrow \delta_{ij} = 0$ are easy:

$$I_{ij} = -\rho_0 \int_{-a/2}^{a/2}\int_{-b/2}^{b/2}\int_{-c/2}^{c/2} x_i x_j \mathrm{d}^3 r = 0$$

because integrals are odd, meaning that

$$\int x_i \mathrm{d}x_i = 0, \quad \forall i.$$

Now we can construct our inertia tensor

$$I = \begin{pmatrix} \frac{M}{12}(b^2+c^2) & 0 & 0 \\ 0 & \frac{M}{12}(a^2+c^2) & 0 \\ 0 & 0 & \frac{M}{12}(a^2+b^2) \end{pmatrix}$$

For a thin rod we have that $b \to 0$ and $c \to 0$ and thus

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{Ma^2}{12} & 0 \\ 0 & 0 & \frac{Ma^2}{12} \end{pmatrix}$$

3. **Solution**:
Components of the moment of inertia tensor can be calculated using:

$$I_{ij} = \int d^3 r \rho (\delta_{ij} r^2 - r_i r_j)$$

Now because our system consists of point masses of mass m, integral is replaced by sum

$$I_{ij} = \sum_{\alpha=1}^{4} m(\delta_{ij} r_\alpha^2 - r_{i,\alpha} r_{j,\alpha}) \,,$$

where $\alpha$ goes through all the mass points. Now using above formula and the coordinates of the masses we get:

$$I_{11} = \sum m(y_\alpha^2 + z_\alpha^2) = m(a^2 + (-a)^2) = 2ma^2$$
$$I_{12} = I_{21} = -\sum m x_\alpha y_\alpha = -m(a^2 + (-a)^2) = -2ma^2$$
$$I_{13} = I_{31} = -\sum m x_\alpha z_\alpha = 0$$
$$I_{22} = \sum m(x_\alpha^2 + z_\alpha^2) = m(a^2 + a^2 + (-a)^2 + (-a)^2) = 4ma^2$$
$$I_{23} = I_{32} = -\sum m y_\alpha z_\alpha = 0$$
$$I_{33} = \sum m(x_\alpha^2 + y_\alpha^2) = 6ma^2$$

Tensor of inertia is then:

$$I = \begin{pmatrix} 2ma^2 & -2ma^2 & 0 \\ -2ma^2 & 4ma^2 & 0 \\ 0 & 0 & 6ma^2 \end{pmatrix}$$

In order to find principal moments of inertia we must find a basis where

$$I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}$$

This can be done like in the lecture notes by solving eigenvalue equation:

$$(I - \lambda 1)a = 0$$

This has a nonzero solution, if:

$$\det(I - \lambda 1) = 0$$

We therefore write:

$$\det \begin{pmatrix} 2ma^2 - \lambda & -2ma^2 & 0 \\ -2ma^2 & 4ma^2 - \lambda & 0 \\ 0 & 0 & 6ma^2 - \lambda \end{pmatrix}$$

Determinant gives us:

$$(6ma^2 - \lambda)(\lambda^2 - 6ma^2\lambda + 4m^2a^4) = 0$$

Solutions to above equation are then the principal moments of inertia:

$$I_{1,2} = (3 \pm \sqrt{5})ma^2$$
$$I_3 = 6ma^2$$

It then remains to be shown that smallest principal moment of inertia corresponds to axis with polar angle $31.7^o$. To do this we need the coordinate transition matrix from exercise 1:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

**Method 1:**

Now we can calculate components of the inertia tensor in primed coordinates. Components then become functions of the polar angle and we can solve the angle by searching the critical points of the function. Lets start with the first component:

$$\begin{aligned} I_{1'1'} &= \sum m(y'^2_\alpha + z'^2_\alpha) = \sum my'^2_\alpha = m\sum(-x_\alpha \sin\phi + y_\alpha \cos\phi)^2 \\ &= \sin^2\phi \sum mx_\alpha^2 + \cos^2\phi \sum my_\alpha^2 - 2\sin\phi\cos\phi \sum mx_\alpha y_\alpha \\ &= \sin^2\phi I_{22} + \cos^2\phi I_{11} + 2\sin\phi\cos\phi I_{12} \end{aligned}$$

where we noticed that summations are actually same ones we encountered while calculating the tensor first time. Next we use following trigonometric identities to make our result look nicer:

$$\cos^2\phi = \frac{1+\cos 2\phi}{2}$$
$$\sin^2\phi = \frac{1-\cos 2\phi}{2}$$
$$2\sin\phi\cos\phi = \sin 2\phi$$

So we get:

$$I_{1'1'} = \frac{I_{11}+I_{22}}{2} + \frac{I_{11}-I_{22}}{2}\cos 2\phi + I_{12}\sin 2\phi$$

Now we derivate this with respect to angle and set it equal to zero:

$$\frac{\partial I_{1'1'}}{\partial \phi} = (I_{22} - I_{11})\sin 2\phi + 2I_{12}\cos 2\phi = 0$$

This then gives us

$$\tan 2\phi = -\frac{2I_{12}}{I_{22} - I_{11}}$$
$$\phi = \frac{1}{2}\tan^{-1} - \frac{2I_{12}}{I_{22} - I_{11}}$$
$$= \frac{1}{2}\tan^{-1}\frac{4ma^2}{(4-2)ma^2} \approx 31.7^o$$

Let's check that this actually gives correct principal moment of inertia by substituting this into our equation:

$$I_{1'1'} = \frac{2ma^2 + 4ma^2}{2} + \frac{2ma^2 - 4ma^2}{2}\cos(2 \times 31.7) - 2ma^2 \sin(2 \times 31.7)$$
$$\approx 0.763932 \approx 3 - \sqrt{5}$$

Smallest pmi is then given by axis with polar angle 31.7°. Lets have a bit more fun and calculate also second pmi in primed coordinates:

$$\begin{aligned} I_{2'2'} &= \sum m(x'^2_i + z'^2_i) = m\sum(x_i \cos\phi + y_i \sin\phi)^2 \\ &= \cos^2\phi \sum mx_i^2 + \sin^2\phi \sum my_i^2 + 2\sin\phi\cos\phi \sum mx_i y_i \\ &= \cos^2\phi I_{22} + \sin^2\phi I_{11} - 2\sin\phi\cos\phi I_{12} \\ &= \frac{I_{11} + I_{22}}{2} + \frac{I_{22} - I_{11}}{2}\cos 2\phi - I_{12}\sin 2\phi \end{aligned}$$

We then make a substitution and get:

$$I_{2'2'} = \frac{2ma^2 + 4ma^2}{2} + \frac{4ma^2 - 2ma^2}{2}\cos(2 \times 31.7) - (-2ma^2)\sin(2 \times 31.7)$$
$$\approx 3 + \sqrt{5}$$

which is exactly what we suspected. We can then safely say that smallest principal moment of inertia corresponds to axis with polar angle 31.7°.

**Method 2:**
We can calculate the eigenvector corresponding to the smallest principal moment of inertia $I_1 = (3 \pm \sqrt{5})ma^2$. This is done by solving the problem

$$(I - I_1 1)\boldsymbol{x} = 0\,.$$

The normalized solution is

$$\boldsymbol{x} = \sqrt{\frac{2}{5 - \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{1}{2}(-1 + \sqrt{5}) \\ 0 \end{pmatrix}.$$

Now we see if this is the same as a rotation of the $x$-axis:

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} = \sqrt{\frac{2}{5 - \sqrt{5}}} \begin{pmatrix} 1 \\ \frac{1}{2}(-1 + \sqrt{5}) \end{pmatrix}.$$

Now

$$\phi = \arccos\left(\sqrt{\frac{2}{5 - \sqrt{5}}}\right) \approx 31.7^{\circ}$$
$$\phi = \arcsin\left(\sqrt{\frac{2}{5 - \sqrt{5}}}\frac{1}{2}(-1 + \sqrt{5})\right) \approx 31.7^{\circ}$$

4. **Solution**:

The kinetic energy is

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + T'$$

where $\mathbf{R}$ is the location of c.m (center of mass), $M$ total mass and $T'$ is the kinetic energy with respect to the c.m coordinate system. On the other hand we can represent the kinetic energy with inertia tensor

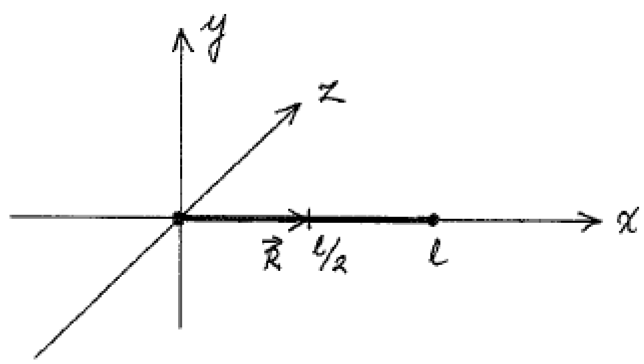
$$\frac{1}{2}\sum_{i=1}^{3} I_i\omega_i^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_{i=1}^{3} I_i'\omega_i^2$$

where $I'$ is inertia tensor in the c.m coordinate system. If $\boldsymbol{\omega} = \omega\mathbf{i}$, then $\dot{\mathbf{R}} = 0$ and $I_1 = I_1'$. If $\boldsymbol{\omega} = \omega\mathbf{j}$, then $\dot{\mathbf{R}} = \frac{l}{2}\omega$ and thus

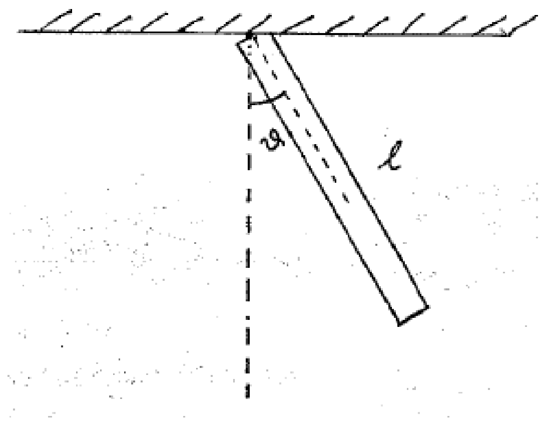
$$I_2 = M\left(\frac{l}{2}\right)^2 + I_2' = \frac{Ml^2}{4} + \frac{Ml^2}{12} = \frac{1}{3}Ml^2.$$

In a similar manner if $\boldsymbol{\omega} = \omega\mathbf{k}$, then

$$I_3 = \frac{1}{3}Ml^2.$$



5. **Solution**:




Let's choose the axis such that the rod is rotating around the y-axis:

$$\boldsymbol{\omega} = \begin{pmatrix} 0 \\ \omega \\ 0 \end{pmatrix} \qquad \omega = \dot{\theta}.$$

We have already calculated the inertia tensor

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3}ml^2 & 0 \\ 0 & 0 & \frac{1}{3}ml^2 \end{pmatrix}.$$

So the kinetic energy is

$$T = \frac{1}{2}\sum_{ij} \omega_i I_{ij} \omega_j = \frac{1}{2}\sum_i I_{ii}\omega_i^2 = \frac{1}{6}ml^2\dot{\theta}^2.$$

The potential energy is the the potential energy of the c.m:

$$V = -\int_0^l mg\frac{l'}{l}\cos\theta \mathrm{d}l' = -\frac{1}{2}mgl\cos\theta.$$

Thus the Lagrangian is

$$L = T - V = \frac{1}{6}ml^2\dot{\theta}^2 + \frac{1}{2}mgl\cos\theta.$$

For small oscillations it holds that $|\theta| << 1 \Rightarrow \cos\theta \approx 1 - \frac{1}{2}\theta^2$:

$$L \approx \frac{1}{6}ml^2\dot{\theta}^2 - \frac{1}{4}mgl\theta^2 + \text{constant}.$$

Now the Lagrange's equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Leftrightarrow$$

$$\frac{1}{3}ml^2\ddot{\theta} + \frac{1}{2}mgl\theta = 0$$

$$\Leftrightarrow$$

$$\ddot{\theta} + \frac{3}{2}\frac{g}{l}\theta = 0$$

and its solution is

$$\theta = A\cos(\Omega t + \delta)$$

where $A$ and $\delta$ are constants and the angular velocity is

$$\Omega = \sqrt{\frac{3}{2}\frac{g}{l}}.$$