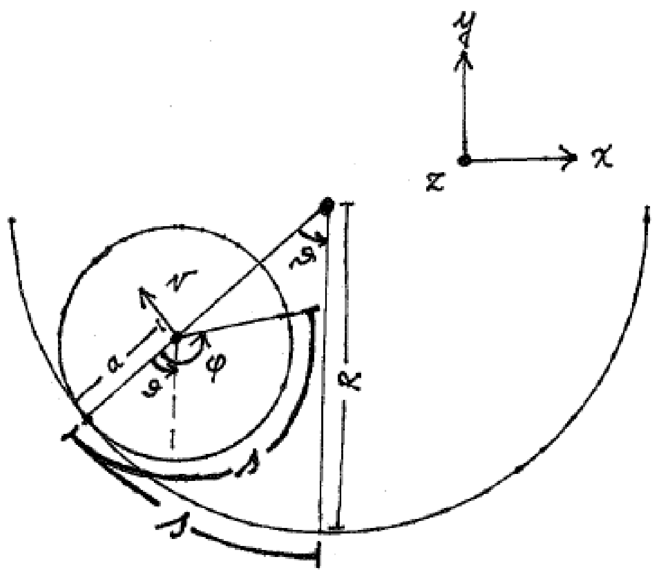
1. **Solution**:



The total kinetic energy consists of the kinetic energy for the c.m (center of mass) and the kinetic energy related to the rotation around the c.m

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2$$

where ω is the angular velocity of the cylinder around its axis and I is the moment of inertia for the cylinder is $1/2Ma^2$. The c.m has velocity

$$v = (R - a)\dot{\theta}.$$

The rotating angle of the cylinder has to be calculated with respect to $-y$-axis counterclockwise around z-axis. Thus the angle is $\alpha = \varphi - \theta$ and the angular velocity $\omega = \dot{\alpha} = \dot{\varphi} - \dot{\theta}$. Now we solve the angle φ:

$$\begin{cases} s = R\theta \\ s = a\varphi \end{cases} \Rightarrow a\varphi = R\theta \Rightarrow \varphi = \frac{R}{a}\theta \Rightarrow \dot{\varphi} = \frac{R}{a}\dot{\theta}.$$

This means

$$\omega = \dot{\varphi} - \dot{\theta} = \frac{R}{a}\dot{\theta} - \dot{\theta} = \frac{R-a}{a}\dot{\theta}.$$

Combining our results leads to kinetic energy

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}M[(R-a)\dot{\theta}]^2 + \frac{1}{2} * \frac{1}{2}Ma^2\left(\frac{R-a}{a}\dot{\theta}\right)^2 \\ &= \frac{1}{2}M(R-a)^2\dot{\theta}^2 + \frac{1}{4}M(R-a)^2\dot{\theta}^2 \\ &= \frac{3}{4}M(R-a)^2\dot{\theta}^2. \end{aligned}$$

The c.m. has potential

$$V = -Mg(R-a)\cos\theta.$$

The Lagrangian of the system is

$$\begin{aligned} L &= \frac{3}{4}M(R-a)^2\dot{\theta}^2 + Mg(R-a)\cos\theta \\ &\approx \frac{3}{4}M(R-a)^2\dot{\theta}^2 - \frac{1}{2}Mg(R-a)\theta^2 + Mg(R-a) \end{aligned}$$

where we used an approximation for small oscillations $|\theta| << 1$, and after applying the Lagrange equation we get the equation of motion:

$$\begin{aligned} &\frac{3}{2}M(R-a)^2\ddot{\theta} + Mg(R-a)\theta = 0 \\ &\Leftrightarrow \\ &\ddot{\theta} + \frac{2g}{3(R-a)}\theta = 0 \end{aligned}$$

that is the equation of the harmonic oscillator with a frequency

$$\Omega = \sqrt{\frac{2}{3}\frac{g}{(R-a)}}.$$

This result has a very interesting consequence

$$\lim_{a\to R}\Omega = \lim_{a\to R}\sqrt{\frac{2}{3}\frac{g}{(R-a)}} = \infty.$$

2. **Solution**:
   From the lectures we know

$$T = \frac{1}{2}I_1(\dot{\beta}\sin\gamma - \dot{\alpha}\sin\beta\cos\gamma)^2 + \frac{1}{2}I_2(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma)^2 + \frac{1}{2}I_3(\dot{\alpha}\cos\beta + \dot{\gamma})^2.$$

The needed derivatives are

$$\frac{\partial T}{\partial\dot{\gamma}} = I_3(\dot{\alpha}\cos\beta + \dot{\gamma}) = \omega_3 I_3$$

and

$$\begin{aligned} \frac{\partial T}{\partial\gamma} &= I_1(\dot{\beta}\sin\gamma - \dot{\alpha}\sin\beta\cos\gamma)(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma) \\ &+ I_2(\dot{\alpha}\sin\beta\sin\gamma + \dot{\beta}\cos\gamma)(\dot{\alpha}\sin\beta\cos\gamma - \dot{\beta}\sin\gamma) \\ &= I_1\omega_1\omega_2 - I_2\omega_1\omega_2 \\ &= (I_1 - I_2)\omega_1\omega_2. \end{aligned}$$

Thus we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\gamma}}\right) - \frac{\partial T}{\partial \gamma} = N_3$$
$$\Leftrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_3 I_3) - (I_1 - I_2)\omega_1\omega_2 = N_3$$
$$\Leftrightarrow$$
$$I_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + N_3.$$

3. **Solution**:
The Lagrangian is

$$L = \frac{1}{2}I_1(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2) + \frac{1}{2}I_3(\dot{\alpha}\cos\beta + \dot{\gamma})^2 - Mgl\cos\beta$$

and the canonical momenta are

$$p_\alpha = \frac{\partial L}{\partial \dot{\alpha}} = I_1 \sin^2\beta\dot{\alpha} + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta$$

and

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3(\dot{\alpha}\cos\beta + \dot{\gamma}).$$

Note that α and γ are cyclic coordinates and thus the momenta p_α and p_γ are conserved. The coordinate β is not cyclic and hence the momentum

$$p_\beta = \frac{\partial L}{\partial \dot{\beta}} = I_1\dot{\beta}$$

is not constant. Now we solve $\dot{\alpha}$ and $\dot{\gamma}$ from the momenta p_α and p_γ:

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} = I_3(\dot{\alpha}\cos\beta + \dot{\gamma})$$
$$\Rightarrow$$
$$\dot{\gamma} = \frac{p_\gamma}{I_3} - \dot{\alpha}\cos\beta$$
$$\Rightarrow$$
$$\dot{\alpha} = \frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}$$

and we put the solved $\dot{\alpha}$ back to $\dot{\gamma}$

$$\dot{\gamma} = \frac{p_\gamma}{I_3} - \dot{\alpha}\cos\beta = \frac{p_\gamma}{I_3} - \frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\cos\beta.$$

The Hamiltonian is

$$\begin{aligned}
H &= \sum_i \dot{q}_i p_i - L \\
&= \dot{\alpha} p_\alpha + \dot{\beta} p_\beta + \dot{\gamma} p_\gamma - L \\
&= \dot{\alpha}(I_1 \sin^2\beta\dot{\alpha} + I_3(\dot{\alpha}\cos\beta + \dot{\gamma})\cos\beta) + \dot{\beta} I_1 \dot{\beta} + I_3\dot{\gamma}(\dot{\alpha}\cos\beta + \dot{\gamma}) \\
&- \left[\frac{1}{2} I_1(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3(\dot{\alpha}\cos\beta + \dot{\gamma})^2 - Mgl\cos\beta\right] \\
&= \frac{1}{2} I_1(\dot{\alpha}^2\sin^2\beta + \dot{\beta}^2) + \frac{1}{2} I_3(\dot{\alpha}\cos\beta + \dot{\gamma})^2 + Mgl\cos\beta \\
&= \frac{1}{2} I_1\left[\left(\frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\right)^2 \sin^2\beta + \dot{\beta}^2\right] + \frac{1}{2} I_3\left[\left(\frac{p_\alpha - p_\gamma\cos\beta}{I_1\sin^2\beta}\right)\cos\beta + \frac{p_\gamma}{I_3} - \dot{\alpha}\cos\beta\right]^2 \\
&+ Mgl\cos\beta \\
&= \frac{p_\alpha^2 - p_\alpha p_\gamma\cos\beta}{I_1\sin^2\beta} + \frac{1}{2} I_1\dot{\beta}^2 + \frac{p_\alpha^2}{I_3} - \frac{p_\alpha p_\gamma - p_\alpha^2\cos\beta}{I_1\sin^2\beta}\cos\beta - \frac{1}{2} I_1 \frac{p_\alpha^2}{I_1^2\sin^2\beta} \\
&+ I_1\frac{p_\alpha p_\gamma}{I_1^2\sin^2\beta}\cos\beta - \frac{1}{2} I_1\frac{p_\gamma^2\cos^2\beta}{I_1\sin^2\beta} - \frac{1}{2}\frac{p_\gamma^2}{I_3} + Mgl\cos\beta \\
&= \frac{1}{2}\frac{p_\gamma^2}{I_1^2\sin^2\beta} + \frac{1}{2} I_1\dot{\beta}^2 + \frac{1}{2}\frac{p_\gamma}{I_3} - \frac{p_\alpha p_\gamma}{I_1\sin^2\beta}\cos\beta + \frac{1}{2}\frac{p_\gamma^2\cos^2\beta}{I_1\sin^2\beta} + Mgl\cos\beta \\
&= \frac{1}{2} I_1\dot{\beta}^2 + \underbrace{\frac{1}{2}\frac{p_\gamma^2}{I_3} + \frac{1}{2}\frac{(p_\alpha - p_\gamma\cos\beta)^2}{I_1\sin^2\beta} + Mgl\cos\beta}_{=V_{\text{eff}}(\beta)} \\
&= \frac{1}{2} I_1\dot{\beta}^2 + V_{\text{eff}}(\beta)
\end{aligned}$$

Because $\partial_t L = 0$, the Hamiltonian is a constant of motion. The Hamiltonian is same as the total energy of the system $H = E$:

$$\begin{aligned}
&E = \frac{1}{2} I_1\dot{\beta}^2 + V_{\text{eff}}(\beta) \\
&\Leftrightarrow \\
&\Bigg(\underbrace{\frac{\mathrm{d}\beta}{\mathrm{d}t}}_{=\dot{\beta}}\Bigg)^2 = \frac{2}{I_1}(E - V_{\text{eff}}) \\
&\Leftrightarrow \\
&\frac{\mathrm{d}\beta}{\mathrm{d}t} = \pm\sqrt{\frac{2}{I_1}(E - V_{\text{eff}})} \\
&\Leftrightarrow \\
&\mathrm{d}t = \pm\sqrt{\frac{I_1}{2}}\frac{\mathrm{d}\beta}{\sqrt{E - V_{\text{eff}}}} \\
&\Leftrightarrow \\
&t = \pm\sqrt{\frac{I_1}{2}}\int\frac{\mathrm{d}\beta}{\sqrt{E - V_{\text{eff}}}}.
\end{aligned}$$

4. **Solution**:
The pendulum has the Lagrangian

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl \cos\theta.$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \Rightarrow \dot{\theta} = \frac{p}{ml^2}.$$

Thus the Hamiltonian is

$$\begin{aligned} H &= \dot{\theta} p - L \\ &= \frac{p^2}{ml^2} - \frac{1}{2} ml^2 \left(\frac{p}{ml^2} \right)^2 - mgl \cos\theta \\ &= \frac{p^2}{2ml^2} - mgl \cos\theta. \end{aligned}$$

and the Hamilton equations are

$$\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p} = \frac{p}{ml^2} \\ \dot{p} &= -\frac{\partial H}{\partial \theta} = -mg \sin\theta. \end{aligned}$$

From the Hamilton equations we get

$$\begin{aligned} &\dot{\theta} = \frac{p}{ml^2} \\ &\Rightarrow \\ &\ddot{\theta} = \frac{\dot{p}}{ml^2} = -\frac{mgl}{ml^2} \sin\theta \\ &\Leftrightarrow \\ &\ddot{\theta} + \frac{g}{l} \sin\theta = 0 \end{aligned}$$

that is the same equation of motion as in the Lagrange's formalism, and in the limit of small oscillations it reduces to the equation for the harmonic oscillation.