1. **Solution**:
   The proof is

$$
\begin{aligned}
[AB, C]_{\mathrm{PB}} &= \sum_i \left(\frac{\partial AB}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial C}{\partial q_i}\frac{\partial AB}{\partial p_i} \right) \\
&= \sum_i \left[\left(\frac{\partial A}{\partial q_i} B + \frac{\partial B}{\partial q_i} A \right) \frac{\partial C}{\partial p_i} - \frac{\partial C}{\partial q_i} \left(\frac{\partial A}{\partial p_i} B + \frac{\partial B}{\partial p_i} A \right) \right] \\
&= \sum_i \left(A\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - A\frac{\partial C}{\partial q_i}\frac{\partial B}{\partial p_i} + B\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - B\frac{\partial C}{\partial q_i}\frac{\partial A}{\partial p_i} \right) \\
&= A\sum_i \left(\frac{\partial B}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial C}{\partial q_i}\frac{\partial B}{\partial p_i} \right) + B\sum_i \left(\frac{\partial A}{\partial q_i}\frac{\partial C}{\partial p_i} - \frac{\partial C}{\partial q_i}\frac{\partial A}{\partial p_i} \right) \\
&= A[B, C]_{\mathrm{PB}} + [A, C]_{\mathrm{PB}} B.
\end{aligned}
$$

2. **Solution**:
   a)
   If the Hamiltonian does not explicitly depend on time i.e.

$$\frac{\partial H}{\partial t} = 0,$$

then it holds

$$\frac{\mathrm{d}H}{\mathrm{d}t} = -[H, H]_{\mathrm{PB}} + \underbrace{\frac{\partial H}{\partial t}}_{=0} = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} \right) = 0.$$

b)
If the Hamiltonian has a property

$$\frac{\partial H}{\partial q_k} = 0,$$

then it holds that

$$
\begin{aligned}
\frac{\mathrm{d}p_k}{\mathrm{d}t} &= -[H, p_k]_{\mathrm{PB}} + \underbrace{\frac{\partial p_k}{\partial t}}_{=0} \\
&= -\sum_i \left(\frac{\partial H}{\partial q_i} \underbrace{\frac{\partial p_k}{\partial p_i}}_{=\delta_{ki}} - \underbrace{\frac{\partial p_k}{\partial q_i}}_{=0} \frac{\partial H}{\partial p_i} \right) \\
&= -\sum_i \delta_{ik} \frac{\partial H}{\partial q_i} \\
&= -\frac{\partial H}{\partial q_k} \\
&= 0.
\end{aligned}
$$

3. **Solution**:
Our statement is

$$[L_i, L_j]_{\text{PB}} = \sum_k \epsilon_{ijk}.$$

If $i = j$, then $[L_i, L_i]_{\text{PB}} = 0$ and $\epsilon_{iik} = 0$ because

$$\epsilon_{ijk} = -\epsilon_{jik} \Rightarrow \epsilon_{iik} = -\epsilon_{iik} \Rightarrow \epsilon_{iik} = 0.$$

Hence the statement holds. Furthermore, the statement is correct if we change the indices $(i, j) \to (j, i)$, because $[L_i, L_i]_{\text{PB}} = -[L_j, L_i]_{\text{PB}}$ and $\epsilon_{ijk} = -\epsilon_{jik}$. Thus we have left only three cases to prove: $(1, 2)$, $(1, 3)$ and $(2, 3)$. We recall

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = (yp_z - zp_y)\mathbf{i} + (zp_x - xp_z)\mathbf{j} + (xp_y - yp_x)\mathbf{k} = L_1\mathbf{i} + L_2\mathbf{j} + L_3\mathbf{k}.$$

If $(i, j) = (1, 2)$, then

$$\begin{aligned}
[L_1, L_2]_{\text{PB}} &= \sum_i \left[\frac{\partial}{\partial q_i}(yp_z - zp_y)\frac{\partial}{\partial p_i}(zp_x - xp_z) - \frac{\partial}{\partial q_i}(zp_x - xp_z)\frac{\partial}{\partial p_i}(yp_z - zp_y) \right] \\
&= \sum_i [(\delta_{i2}p_z - \delta_{i3}p_y)(z\delta_{i1} - x\delta_{i3}) - (\delta_{i3}p_x - \delta_{i1}p_z)(y\delta_{i3} - z\delta_{i2})] \\
&= \sum_i [\delta_{i3}^2 p_y x - \delta_{i3}^2 p_x y] \\
&= xp_y - yp_x \\
&= L_3
\end{aligned}$$

(note that terms e.g. $\delta_{i1}\delta_{i2}$, $\delta_{i1}\delta_{i3}$ ... sum up to zero) and the permutation symbol gives

$$\sum_k \epsilon_{12k} = \underbrace{\epsilon_{121}}_{=0} L_1 + \underbrace{\epsilon_{122}}_{=0} L_2 + \underbrace{\epsilon_{123}}_{=1} L_3 = L_3.$$

Others in a similar way:

$$\begin{aligned}
[L_1, L_3]_{\text{PB}} &= \sum_i \left[\frac{\partial}{\partial q_i}(yp_z - zp_y)\frac{\partial}{\partial p_i}(xp_y - yp_x) - \frac{\partial}{\partial q_i}(xp_y - yp_x)\frac{\partial}{\partial p_i}(yp_z - zp_y) \right] \\
&= \sum_i [(\delta_{i2}p_z - \delta_{i3}p_y)(x\delta_{i2} - y\delta_{i1}) - (\delta_{i1}p_y - \delta_{i2}p_x)(y\delta_{i3} - z\delta_{i2})] \\
&= \sum_i [\delta_{i2}^2 p_z x - \delta_{i2}^2 p_x z] \\
&= xp_z - zp_x \\
&= -L_2
\end{aligned}$$

and

$$\sum_k \epsilon_{13k} = \underbrace{\epsilon_{131}}_{=0} L_1 + \underbrace{\epsilon_{132}}_{=-1} L_2 + \underbrace{\epsilon_{133}}_{=0} L_3 = -L_2$$

and the last case

$$
\begin{aligned}
[L_2, L_3]_{\mathrm{PB}} &= \sum_i \left[\frac{\partial}{\partial q_i}(zp_x - xp_z)\frac{\partial}{\partial p_i}(xp_y - yp_x) - \frac{\partial}{\partial q_i}(xp_y - yp_x)\frac{\partial}{\partial p_i}(zp_x - xp_z)\right] \\
&= \sum_i [(\delta_{i3}p_x - \delta_{i1}p_x)(x\delta_{i2} - y\delta_{i1}) - (\delta_{i1}p_y - \delta_{i2}p_x)(z\delta_{i1} - x\delta_{i3})] \\
&= \sum_i [\delta_{i1}^2 p_z y - \delta_{i1}^2 p_y z] \\
&= yp_z - zp_y \\
&= L_1
\end{aligned}
$$

and

$$
\sum_k \epsilon_{23k} = \underbrace{\epsilon_{231}}_{=1} L_1 + \underbrace{\epsilon_{232}}_{=0} L_2 + \underbrace{\epsilon_{233}}_{=0} L_3 = L_1.
$$

Now we see that the statement holds for every index. Because

$$
L^2 = \sum_j L_j^2,
$$

direct calculation gives us

$$
\begin{aligned}
[L^2, L_i]_{\mathrm{PB}} &= \sum_j [L_j^2, L_i]_{\mathrm{PB}} \\
&= \sum_j \Big(L_j [L_j, L_i]_{\mathrm{PB}} + [L_j, L_i]_{\mathrm{PB}} L_j \Big) \\
&= 2 \sum_j L_j [L_j, L_i]_{\mathrm{PB}} \\
&= 2 \sum_j L_j \sum_k \epsilon_{jik} L_k \\
&= 2 \sum_{jk} \epsilon_{jik} L_j L_k \\
&= 0,
\end{aligned}
$$

where we noticed

$$
\begin{aligned}
&\sum_{jk} \epsilon_{jik} L_j L_k = -\sum_{jk} \epsilon_{kij} L_k L_j = -\sum_{kj} \epsilon_{jik} L_j L_k =^{(*)} -\sum_{jk} \epsilon_{jik} L_j L_k \\
&\Rightarrow \sum_{jk} \epsilon_{jik} L_j L_k = 0.
\end{aligned}
$$

In the step $(*)$ we renamed the indices $(j, k) \to (k, j)$.
If we consider a group of particles (N particles) that has coordinates $q_i^{(a)}$ and momenta $p_i^{(a)}$ with indices $a = 1, \ldots, N$ and $i = 1, 2, 3$, we have

$$
\mathbf{L} = \sum_{a=1}^{N} \mathbf{L}^{(a)},
$$

where

$$\mathbf{L}^{(a)} = \mathbf{r}^{(a)} \times \mathbf{p}^{(a)}.$$

Now we calculate

$$\begin{aligned}
[L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}} &= \sum_{c=1}^{N} \sum_{k=1}^{3} \left(\frac{\partial L_i^{(a)}}{\partial q_k^{(c)}} \frac{\partial L_j^{(b)}}{\partial p_k^{(c)}} - \frac{\partial L_j^{(b)}}{\partial q_k^{(c)}} \frac{\partial L_i^{(a)}}{\partial p_k^{(c)}} \right) \\
&= \sum_{c} \sum_{k} \left(\delta_{ac} \frac{\partial L_i^{(a)}}{\partial q_k^{(a)}} \delta_{bc} \frac{\partial L_j^{(b)}}{\partial p_k^{(b)}} - \delta_{bc} \frac{\partial L_j^{(b)}}{\partial q_k^{(b)}} \delta_{ac} \frac{\partial L_i^{(a)}}{\partial p_k^{(a)}} \right) \\
&= \underbrace{\left(\sum_{c} \delta_{ac} \delta_{bc} \right)}_{=\delta_{ab}} \underbrace{\sum_{k} \left(\frac{\partial L_i^{(a)}}{\partial q_k^{(a)}} \frac{\partial L_j^{(b)}}{\partial p_k^{(b)}} - \frac{\partial L_j^{(b)}}{\partial q_k^{(b)}} \frac{\partial L_i^{(a)}}{\partial p_k^{(a)}} \right)}_{=[L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}}} \\
&= \delta_{ab} [L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}}. (*)
\end{aligned}$$

Because $\mathbf{L}^{(a)}$ only depend on $q_i^{(a)}$ and $p_i^{(a)}$ but not on $q_i^{(b)}$ or $p_i^{(b)}$ $(a \neq b)$, it should be

$$[L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}} = 0 \quad a \neq b.$$

The above calculation proves this and is the reason why the many-particle problem reduces to the single particle case:

$$\begin{aligned}
[L_i, L_j]_{\mathrm{PB}} &= \left[\sum_a L_i^{(a)}, \sum_b L_j^{(b)} \right]_{\mathrm{PB}} \\
&= \sum_{ab} \underbrace{[L_i^{(a)}, L_J^{(b)}]_{\mathrm{PB}}}_{=\delta_{ab}[L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}}} \\
&= \sum_{ab} \delta_{ab} [L_i^{(a)}, L_j^{(b)}]_{\mathrm{PB}} \\
&= \sum_{a} \underbrace{[L_i^{(a)}, L_j^{(a)}]_{\mathrm{PB}}}_{=\sum_k \epsilon_{ijk} L_k^{(a)}} \\
&= \sum_{a} \sum_{k} \epsilon_{ijk} L_k^{(a)} \\
&= \sum_{k} \epsilon_{ijk} \underbrace{(\sum_{a} L_k^{(a)})}_{=L_k} \\
&= \sum_{k} \epsilon_{ijk} L_k.
\end{aligned}$$

Because we already proved $[L^2, L_i]_{\mathrm{PB}}$ for a single particle only using the result $[L_i, L_j]_{\mathrm{PB}} = \sum_k \epsilon_{ijk} L_k$, we can prove the same result to a many-particle case reducing the Poisson brackets to a single particle case using the formula $(*)$ and using result $[L_i, L_j]_{\mathrm{PB}} = \sum_k \epsilon_{ijk} L_k$.

4. **Solution**:
The Lagrangian is

$$L = T = \frac{1}{2} m \dot{x}^2$$

meaning that canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$$

and thus the Hamiltonian has a form

$$H(x,p) = p\dot{x} - L = m\dot{x}^2 - \frac{1}{2} m\dot{x}^2 = \frac{1}{2} m\dot{x}^2 = \frac{p^2}{2m}.$$

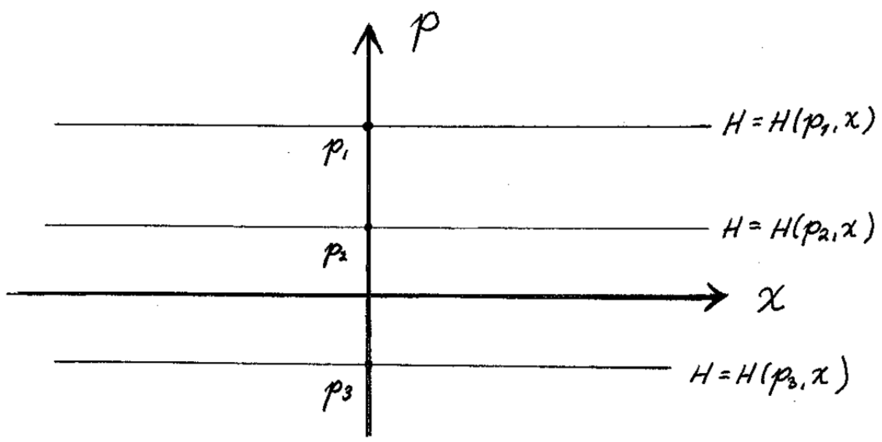
The Hamilton equations say that

$$\dot{x} = \frac{\partial H}{\partial p} \Rightarrow \dot{x} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial x} \Rightarrow \dot{p} = 0$$

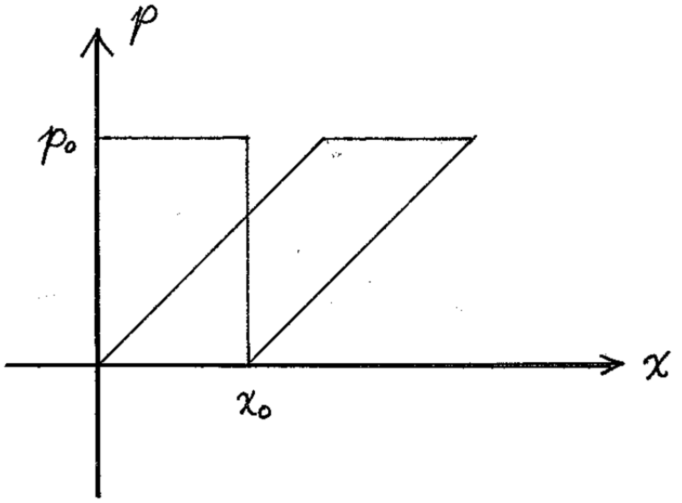
and the solution for these equations is

$$p(t) = p(0)$$
$$x(t) = \frac{p(0)}{m} t + x(0).$$

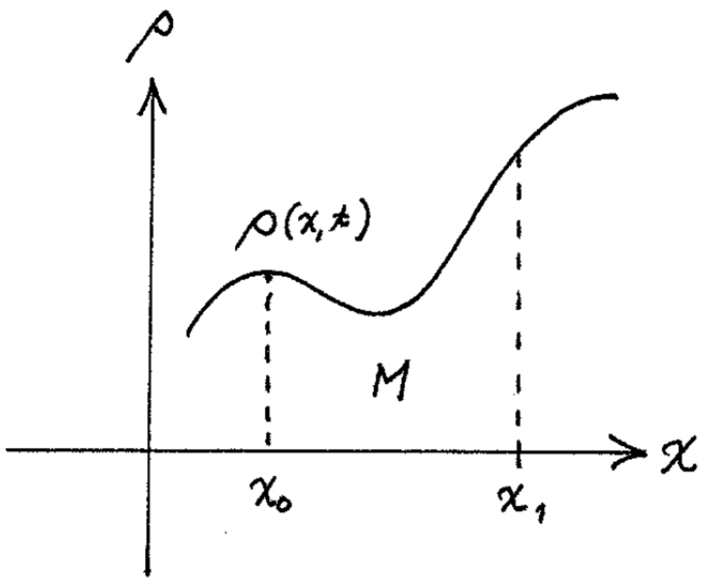
Because L does not explicitly depend on time, the Hamiltonian is a constant of motion and that is the reason why the solutions of the Hamilton equations are contours for the Hamiltonian. Because the Hamiltonian just depends on the momentum, the contours of the Hamiltonian are parallel to the x-axis.



The solution says that a point is moving with speed proportional to the initial value of momentum along the line $p = p(0)$. This implies that the points in the base of the rectangle ($p = 0$) are staying still and the points in the top are moving along x-axis with constant speed $p(0)/m$. It is easy to see that the initial rectangle changes to a parallelogram when time goes on. With a short calculation using the solution, we can notice that the area A is the same in the rectangle as in the parallelogram. The conservation of the area means that also density is constant and thus Liouville's theorem holds.



5. **Solution**:
We can consider an arbitrary distribution $\rho(x, t)$. Let's denote an arbitrary interval with $I = [x_0, x_1]$ that does not depend on time. The mass



M inside the interval I is

$$M = \int_I \rho(x,t)\mathrm{d}x.$$

The change of mass with time is

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_I \rho(x,t)\mathrm{d}x = \int_I \frac{\partial\rho}{\partial t}(x,t)\mathrm{d}x.$$

On the other hand, there is mass flow going in and coming out in the ending points of the interval. In the point x the mass flux to the positive x-direction is

$$\rho(x,t)v(x,t).$$

Now we deduce that the mass change is the same as the net flux:

$$\begin{aligned}
\frac{\mathrm{d}M}{\mathrm{d}t} &= \rho(x_0,t)v(x_0,t) - \rho(x_1,t)v(x_1,t) \\
&= -\bigg/_{x_0}^{x_1} \rho(x,t)v(x,t) \\
&= -\int_{x_0}^{x_1} \frac{\mathrm{d}}{\mathrm{d}x}[\rho(x,t)v(x,t)]\mathrm{d}x.
\end{aligned}$$

Because our two representations for the mass change have to be same, we have a relation

$$\begin{aligned}
&\int_I \frac{\partial\rho}{\partial t}(x,t)\mathrm{d}x = -\int_I \frac{\mathrm{d}}{\mathrm{d}x}[\rho(x,t)v(x,t)]\mathrm{d}x \\
&\Leftrightarrow \\
&\int_I \left\{\frac{\partial\rho}{\partial t}(x,t) + \frac{\mathrm{d}}{\mathrm{d}x}[\rho(x,t)v(x,t)]\right\} = 0
\end{aligned}$$

Because our interval I is arbitrary, the integrand must always be zero i.e.

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}(x,t) + \frac{\mathrm{d}}{\mathrm{d}x}[\rho(x,t)v(x,t)] = 0 \\
&\Leftrightarrow \\
&\frac{\partial\rho}{\partial t}(x,t) + v(x,t)\frac{\mathrm{d}\rho}{\mathrm{d}x}(x,t) + \rho(x,t)\frac{\mathrm{d}v}{\mathrm{d}x}(x,t) = 0 \\
&\Leftrightarrow \\
&\frac{\partial\rho}{\partial t}(x,t) + v(x,t)\frac{\mathrm{d}\rho}{\mathrm{d}x}(x,t) = -\rho(x,t)\frac{\mathrm{d}v}{\mathrm{d}x}(x,t).
\end{aligned}$$