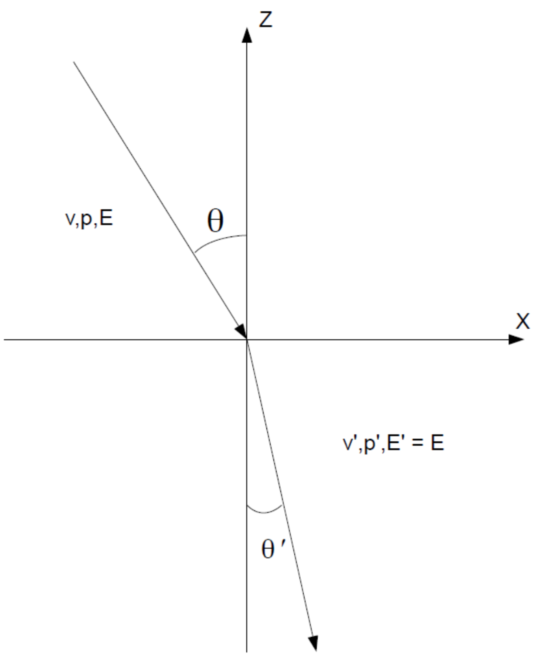
1. **Solution**:



We can choose without losing the generality that the particle is moving in the xz-plane, so $v_y = 0$.
First we use the conservation law of momentum in x-direction:

$$p \sin\theta = p' \sin\theta' \Rightarrow \frac{p'}{p} = \frac{\sin\theta}{\sin\theta'}.$$

Then we use the conservation law of energy:

$$\frac{p^2}{2m} + V = \frac{p'^2}{2m} + V' = E, \quad \text{because } E = E'$$

$$\Leftrightarrow$$

$$\begin{cases} \frac{p^2}{2m} + V = E \\ \frac{p'^2}{2m} + V' = E \end{cases}$$

$$\Rightarrow$$

$$\left(\frac{p'}{p}\right)^2 = \frac{E - V'}{E - V}$$

$$\Leftrightarrow$$

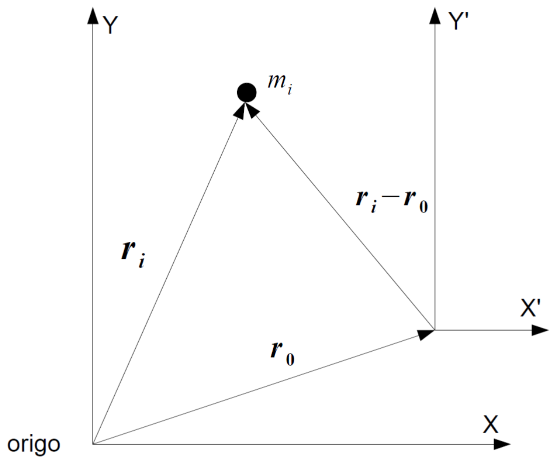
$$\frac{p'}{p} = \sqrt{\frac{E - V'}{E - V}}.$$

Now by combining the conservation laws one gets

$$\frac{\sin\theta}{\sin\theta'} = \frac{p'}{p} = \sqrt{\frac{E-V'}{E-V}} =^{\text{def.}} \frac{n'}{n}$$
$$\Leftrightarrow$$
$$n\sin\theta = n'\sin\theta'.$$

It is interesting to note that in this mechanical picture n is directly proportional to the velocity of the particle, $n \propto v$. This should be contrasted to wave optics, where $n = c/v$, where c is the velocity of light in vacuum and v in the medium. Thus in wave optics n is *inversely* proportional to the velocity of the wave.

2. **Solution**:



Before the actual proof, let's check some relations: firstly

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\underbrace{\sum_i m_i}_{=M}} \Rightarrow M\mathbf{R} = \sum_i m_i \mathbf{r}_i$$

and secondly from the lectures

$$\dot{\mathbf{p}}_i = \mathbf{F}_i^{(\mathrm{e})} + \mathbf{F}_i^{(\mathrm{i})}$$

and thus

$$\begin{aligned}\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \dot{\mathbf{p}}_i &= \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\mathbf{F}_i^{(\mathrm{e})} + \mathbf{F}_i^{(\mathrm{i})}) \\ &= \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{(\mathrm{i})} + \underbrace{\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \mathbf{F}_i^{(\mathrm{e})}}_{=^{\text{def.}}\mathbf{N}'} \\ &= \mathbf{N}' + \underbrace{\sum_i \mathbf{r}_i \times \mathbf{F}_i^{(\mathrm{i})}}_{=0\,\text{look at the lectures}} - \mathbf{r}_0 \times \underbrace{\sum_i \mathbf{F}_i^{(\mathrm{i})}}_{=0\,\text{Newton's 3rd law}} \\ &= \mathbf{N}'.\end{aligned}$$

Now the required proof:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{L}' &= \sum_i \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{r}_i - \mathbf{r}_0)] \times (\mathbf{p}_i - m_i\dot{\mathbf{r}}_0) + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \frac{\mathrm{d}}{\mathrm{d}t}[(\mathbf{p}_i - m_i\dot{\mathbf{r}}_0)] \\
&= \sum_i (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times (\mathbf{p}_i - m_i\dot{\mathbf{r}}_0) + \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times (\dot{\mathbf{p}}_i - m_i\ddot{\mathbf{r}}_0) \\
&= \sum_i m_i \underbrace{(\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0) \times (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_0)}_{=0} + \underbrace{\sum_i (\mathbf{r}_i - \mathbf{r}_0) \times \dot{\mathbf{p}}_i}_{=\mathbf{N}'} - \sum_i (\mathbf{r}_i - \mathbf{r}_0) \times m_i\ddot{\mathbf{r}}_0 \\
&= \mathbf{N}' + \Big(\underbrace{\sum_i m_i\mathbf{r}_0}_{=M\mathbf{r}_0} - \underbrace{\sum_i m_i\mathbf{r}_i}_{=M\mathbf{R}}\Big) \times \ddot{\mathbf{r}}_0 \\
&= \mathbf{N}' + M(\mathbf{r}_0 - \mathbf{R}) \times \ddot{\mathbf{r}}_0.
\end{aligned}$$

If $\dot{\mathbf{L}}' = \mathbf{N}'$, then it has to be that

1. $\mathbf{r}_0 = \mathbf{R}$, so we are in the C.M. coordinate system
2. $\ddot{\mathbf{r}}_0 \equiv 0$, so our coordinate system is an inertial system (= a system at rest or a steadily moving system)
3. $(\mathbf{R} - \mathbf{r}_0)||\ddot{\mathbf{r}}_0$ which means that the acceleration vector and the vector $(\mathbf{R} - \mathbf{r}_0)$ are parallel.

3. **Solution**:
Our equation of motion is

$$m\ddot{\mathbf{r}} = -mg\hat{y} - k\dot{\mathbf{r}}.$$

Let's substitute $\mathbf{r}(t) \approx \mathbf{r}_0(t) + \mathbf{r}_1(t)$ and thus we get

$$m(\ddot{\mathbf{r}}_0 + \ddot{\mathbf{r}}_1) = -mg\hat{y} - k(\dot{\mathbf{r}}_0 + \dot{\mathbf{r}}_1).$$

Now we match the zeroth- and the first-order terms in k:

$$\begin{cases} m\ddot{\mathbf{r}}_0 = -mg\hat{y} \\ m\ddot{\mathbf{r}}_1 = -k\dot{\mathbf{r}}_0. \end{cases}$$

The initial values are

$$\begin{cases} \mathbf{r}_0(0) = \mathbf{r}(0) = 0 \\ \dot{\mathbf{r}}_0(0) = \dot{\mathbf{r}}(0) = \mathbf{v} \end{cases}$$

and because the initial values are zeroth order, it holds $\dot{\mathbf{r}}_i = \mathbf{r}_i = 0\,\forall i > 0$. The solution for the zeroth-oder is easily obtained by integrating twice:

$$\mathbf{r}_0(t) = \mathbf{r}_0(0) + \dot{\mathbf{r}}_0(0)t - \frac{1}{2}gt^2\hat{y}$$

and using the initial values the solution in the component form is

$$\begin{cases} x_0(t) = V_x t \\ y_0 = V_y t - \frac{1}{2} g t^2. \end{cases}$$

The process to solve the first-order equation is

$$\begin{aligned}
\ddot{\mathbf{r}}_1(t) &= -\frac{k}{m}\dot{\mathbf{r}}_0(t) = -\frac{k}{m}(\dot{\mathbf{r}}_0(0) - gt\hat{y}) \\
\Rightarrow \dot{\mathbf{r}}_1(t) &= \int_0^t \ddot{\mathbf{r}}_1(t')\mathrm{d}t' = -\frac{k}{m}\left[\dot{\mathbf{r}}_0(0)t - \frac{1}{2}gt^2\hat{y}\right] \\
\Rightarrow \mathbf{r}_1(t) &= \int_0^t \dot{\mathbf{r}}_1(t')\mathrm{d}t = -\frac{k}{m}\left[\dot{\mathbf{r}}_0(0)\frac{t^2}{2} - \frac{1}{6}gt^3\hat{y}\right] \\
\Leftrightarrow & \begin{cases} x_1(t) = -\frac{k}{2m}V_x t^2 \\ \\ y_1 = -\frac{k}{m}\left[V_y\frac{t^2}{2} - \frac{g}{6}t^3\right]. \end{cases}
\end{aligned}$$

Finally we can write the whole first-order perturbation solution $\mathbf{r}(t) \approx \mathbf{r}_0(t) + \mathbf{r}_1(t)$:

$$\begin{aligned}
x(t) &\approx x_0(t) + x_1(t) = V_x t - \frac{k}{2m}V_x t^2 = V_x\left(t - \frac{k}{2m}t^2\right) \\
y(t) &\approx y_0(t) + y_1(t) = V_y t - \frac{1}{2}gt^2 - \frac{k}{m}\left[V_y\frac{t^2}{2} - \frac{g}{6}t^3\right].
\end{aligned}$$

Now we solve the time for which $y(t) = 0$:

$$\begin{aligned}
& y(t) = V_y t - \frac{1}{2}gt^2 - \frac{k}{m}\left[V_y\frac{t^2}{2} - \frac{g}{6}t^3\right] = 0 \\
& \Leftrightarrow \\
& t = 0 \,\text{or}\, V_y - \frac{1}{2}gt - \frac{k}{m}\left[V_y\frac{t}{2} - \frac{g}{6}t^2\right] = 0.
\end{aligned}$$

We are not interested in the solution $t = 0$ and that is why we are trying to solve the second equation.

Method 1

(recommended)
Let's make again a first-order perturbation but for time $t \approx t_0 + t_1$. Thus the throwing time is

$$\begin{aligned}
& V_y - \left(\frac{1}{2}g + \frac{kV_y}{2m}\right)(t_0 + t_1) + \frac{kg}{6m}(t_0^2 + 2t_0t_1 + t_1^2) = 0 \\
& \Leftrightarrow \\
& V_y - \frac{1}{2}gt_0 = 0 \quad \text{the zeroth order} \\
& -\frac{kV_y}{2m}t_0 - \frac{1}{2}gt_1 + \frac{kg}{6m}t_0^2 = 0 \quad \text{the first order.}
\end{aligned}$$

Remember that we are only interested in the zeroth- and first-order terms and $t_i \propto k^i$. The first-oder term is easy to solve

$$t_0 = \frac{2V_y}{g}$$

and by using the recently solved time t_0 we get

$$
\begin{aligned}
&-\frac{kV_y}{2m}t_0 - \frac{1}{2}gt_1 + \frac{kg}{6m}t_0^2 = 0 \\
&\Leftrightarrow \\
&-\frac{kV_y}{2m}\frac{2V_y}{g} - \frac{1}{2}gt_1 + \frac{kg}{6m}\left(\frac{2V_y}{g}\right)^2 = 0 \\
&\Leftrightarrow \\
&t_1 = -\frac{2}{3}\frac{kV_y^2}{mg^2}.
\end{aligned}
$$

So our solution for the first-order perturbation problem is

$$t \approx t_0 + t_1 = \frac{2V_y}{g} - \frac{2}{3}\frac{kV_y^2}{mg^2}.$$

Method 2

Of course, we can solve the earlier equation exactly

$$V_y - \frac{1}{2}gt - \frac{kV_y}{2m}t + \frac{gk}{6m}t^2 = 0$$

$$t = \frac{3m}{gk}\left[\frac{kV_y}{2m} + \frac{g}{2} \pm \sqrt{\frac{k^2V_y^2}{4m^2} + \frac{gkV_y}{2m} + \frac{g^2}{4} - \frac{2gkV_y}{3m}}\right]$$

$$t = \frac{3m}{gk}\left[\frac{kV_y}{2m} + \frac{g}{2} \pm \frac{g}{2}\sqrt{1 - \frac{2kV_y}{3gm} + \frac{k^2V_y^2}{g^2m^2}}\right]$$

Now it gets a little bit more difficult. We notice that our wanted time is the one with the minus sign in front of the square root. We need only to take the zeroth- and first-order terms and then it is enough to approximate

$$\sqrt{1+x} \approx 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

So taking all the relevant terms gets us to

$$
\begin{aligned}
t &= \frac{3m}{kg}\Big[\frac{g}{2} + \frac{kV_y}{2m} - \frac{g}{2}\Big(1 + \frac{1}{2}(-\frac{2}{3}\frac{kV_y}{mg} + \frac{k^2V_y^2}{g^2m^2}) - \frac{1}{8}(\frac{2}{3}\frac{kV_y}{mg})^2\Big)\Big] \\
&= \frac{3m}{kg}\Big[\frac{2}{3}\frac{kV_y}{m} - \frac{2}{9}\frac{k^2V_y^2}{m^2g}\Big] \\
&= \frac{2V_y}{g} - \frac{2}{3}\frac{kV_y^2}{mg^2}.
\end{aligned}
$$

The result is as before but this method will not always work as method 1 will.
The last thing we calculate is the length of the throwing

$$x = V_x t - \frac{k}{2m} V_x t^2$$

and now we insert the solved time taking only the right orders

$$\begin{aligned} x &= V_x t_0 + V_x t_1 - \frac{k}{2m} V_x t_0^2 \\ &= 2\frac{V_x V_y}{g} - \frac{8}{3}\frac{k}{mg^2} V_x V_y^2. \end{aligned}$$

4. **Solution**:
   a)
   From the potential $U(x)$ we get

$$U(x) = \frac{1}{2}kx^2 - \frac{1}{4}m\epsilon x^4 \Rightarrow F_0(x) = -\nabla U(x) = -kx + \epsilon m x^3$$

and we have an external force

$$F_1(t) = mA\cos(\omega t).$$

Newton's 2nd law gives

$$\dot{\mathbf{p}} = \sum \mathbf{F} \Rightarrow m\ddot{x} = F_0(x) + F_1(t) = -kx + \epsilon m x^3 + mA\cos(\omega t)$$

and using $\omega_0^2 = k/m$ result the equation of motion

$$\ddot{x} = -\omega_0^2 x + \epsilon x^3 + A\cos(\omega t),$$

which is a special case of the Duffing oscillator.
b)
Let's denote $x(t) = x_0(t) + x_1(t) + \cdots$ where $x_k = O(\epsilon^k)$. Because we are only interested up to first-order, so we only put a power series $x(t) = x_0(t) + x_1(t) + O(\epsilon^2)$ to the equation of motion from a):

$$\begin{aligned} \ddot{x} &= \ddot{x}_0 + \ddot{x}_1 + O(\epsilon^2) \\ &= -\omega_0^2(x_0 + x_1 + O(\epsilon^2)) + \epsilon \underbrace{(x_0 + x_1 + O(\epsilon^2))^3}_{=x_0^3 + O(\epsilon)} + A\cos(\omega t) \\ &= -\omega_0^2 x_0 + A\cos(\omega t) - \omega_0^2 x_1 + \epsilon x_0^3 + O(\epsilon^2). \end{aligned}$$

Now we match the zeroth- and first-order terms in ϵ and forget all other corrections:

$$\begin{cases} \ddot{x}_0 = -\omega_0^2 x_0 + A\cos(\omega t) & \text{zeroth order} \\ \ddot{x}_1 = -\omega_0^2 x_1 + \epsilon x_0^3 & \text{first oder.} \end{cases}$$

Now we put the trial solution $x_0 = B\cos(\omega t)$ to the upper equation and assume $\omega^2 \neq \omega_0^2$

$$\begin{aligned}
&\Rightarrow -\omega^2 B\cos(\omega t) = -\omega_0^2 B\cos(\omega t) + A\cos(\omega t)\ \forall t\\
&\Leftrightarrow -\omega^2 B = -\omega_0^2 B + A\\
&\Leftrightarrow B = \frac{A}{\omega_0^2 - \omega^2}
\end{aligned}$$

and for the lower equation we also need $x_1 = C\cos(\omega t) + D\cos(3\omega t)$

$$\begin{aligned}
&\Rightarrow -\omega^2 C\cos(\omega t) - (3\omega)^2 D\cos(3\omega t) = -\omega_0^2[C\cos(\omega t) + D\cos(3\omega t)] + \epsilon B^3\cos^3(\omega t)\\
&\Leftrightarrow -\omega^2 C\cos(\omega t) - 9\omega^2 D\cos(3\omega t) = -\omega_0^2 C\cos(\omega t) - \omega_0^2 D\cos(3\omega t)\\
&+\frac{3}{4}\epsilon B^3\cos(\omega t) + \frac{1}{4}\epsilon B^3\cos(3\omega t)\\
&\Leftrightarrow \begin{cases} -\omega^2 C = -\omega_0^2 C + \frac{3}{4}\epsilon B^3\\ -9\omega^2 D = -\omega_0^2 D + \frac{1}{4}\epsilon B^3\end{cases}\\
&\Leftrightarrow \begin{cases} (\omega_0^2 - \omega^2)C = \frac{3}{4}\epsilon B^3\\ (\omega_0^2 - 9\omega^2)^2 D = \frac{1}{4}\epsilon B^3\end{cases}\\
&\Leftrightarrow^{\text{subst. B}} \begin{cases} C = \frac{3}{4}\frac{A^3}{(\omega_0^2-\omega^2)^4}\epsilon\\ \\ D = \frac{1}{4}\frac{A^3}{(\omega_0^2-\omega^2)^3(\omega_0^2-9\omega^2)}\epsilon.\end{cases}
\end{aligned}$$

Thus the solutions are

$$\begin{aligned}
x_0 &= \frac{A}{\omega_0^2 - \omega^2}\cos(\omega t)\\
x_1 &= \frac{3}{4}\frac{A^3}{(\omega_0^2-\omega^2)^4}\epsilon\cos(\omega t) + \frac{1}{4}\frac{A^3}{(\omega_0^2-\omega^2)^3(\omega_0^2-9\omega^2)}\epsilon\cos(3\omega t).
\end{aligned}$$

The solution of the equation of the motion should be continuous but our approximation diverges in certain points and so the approximation breaks down:

1. $\omega = \omega_0$ the first order term diverges (1:1 resonance)
2. $\omega = \frac{1}{3}\omega_0$ then the external force diverges (overharmonic resonance).

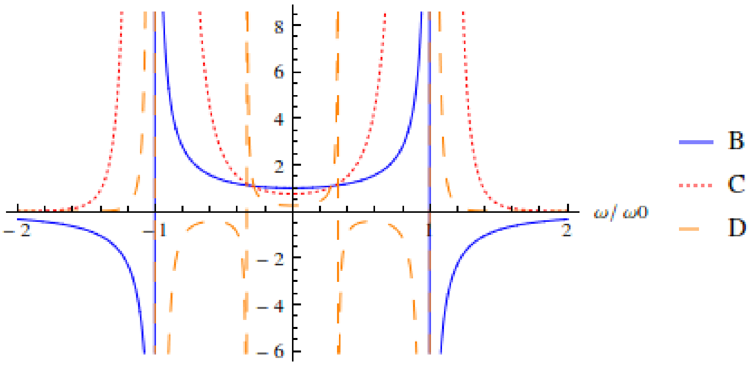


Figure 1: Parameters B (blue),C (dotted red) ja D (dashed orange). $A = 1$, $\omega_0 = 1$.