1. **Solution**:
The velocity in the polar coodinates can be presented as

$$\mathbf{r} = r\hat{r} \Rightarrow \mathbf{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

and thus the kinetic energy in the polar coordinates is

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Denoting the potential with $V(r,\theta)$, we get the Lagrangian

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r,\theta).$$

The partial derivatives are

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \qquad \frac{\partial L}{\partial r} = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \qquad \frac{\partial L}{\partial \theta} = -\frac{\partial V}{\partial \theta}.$$

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{q}_i}\Big) - \frac{\partial L}{\partial q_i} = 0 \qquad q_i = r, \theta$$

the radial equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{r}}\Big) - \frac{\partial L}{\partial r} = 0$$
$$\Leftrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) - mr\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0$$
$$\Leftrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{r}) = mr\dot{\theta}^2 - \frac{\partial V}{\partial r}$$

the angular equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\partial L}{\partial \dot{\theta}}\Big) - \frac{\partial L}{\partial \theta} = 0$$
$$\Leftrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = -\frac{\partial V}{\partial \theta}.$$

Now we notice

$$\begin{aligned}
L_z &= (\mathbf{r} \times m\dot{\mathbf{r}})_z \\
&= m(\mathbf{r} \times \dot{\mathbf{r}})_z \\
&= m[r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})]_z \\
&= m[r\dot{r}\underbrace{(\hat{r} \times \hat{r})}_{=0} + r^2\dot{\theta}\underbrace{(\hat{r} \times \hat{\theta})}_{=\hat{z}}]_z \\
&= (mr^2\dot{\theta}\hat{z})_z \\
&= mr^2\dot{\theta}.
\end{aligned}$$

Furthermore the general force is

$$Q_\theta = \mathbf{F} \cdot \frac{\partial \mathbf{r}}{\partial \theta} = \mathbf{F} \cdot r \underbrace{\frac{\partial \hat{r}}{\partial \theta}}_{=\hat{\theta}} = r\underbrace{(\mathbf{F} \cdot \hat{\theta})}_{=F_\theta} = rF_\theta = N_z,$$

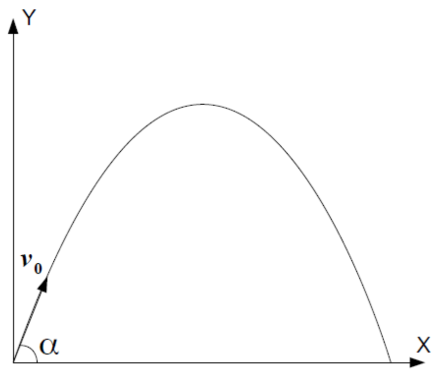
where the last equality comes from the fact that

$$\begin{aligned}
N_z &= (\mathbf{r} \times \mathbf{F})_z \\
&= [r\hat{r} \times (F_r\hat{r} + F_\theta\hat{\theta})]_z, \qquad (\mathbf{F} = F_r\hat{r} + F_\theta\hat{\theta}) \\
&= [rF_r\underbrace{(\hat{r} \times \hat{r})}_{=0} + rF_\theta\underbrace{(\hat{r} \times \hat{\theta})}_{=\hat{z}}]_z \\
&= (rF_\theta\hat{z})_z \\
&= rF_\theta.
\end{aligned}$$

Thus

$$\frac{\mathrm{d}}{\mathrm{d}t}(mr^2\dot{\theta}) = -\frac{\partial V}{\partial \theta} \Rightarrow \dot{L}_z = N_z.$$

2. **Solution**:



a)
Our initial values are

$$\mathbf{r}(0) = 0 \Rightarrow x(0) = y(0) = 0$$
$$\dot{\mathbf{r}}(0) = \mathbf{v}_0 \Rightarrow \begin{cases} \dot{x}(0) = v_0 \cos\alpha \\ \dot{y}(0) = v_0 \sin\alpha. \end{cases}$$

The general coordinates:

$$q_1 \equiv x \qquad \text{and} \qquad q_2 \equiv y,$$

the kinetic energy

$$T = \frac{1}{2} m \dot{\mathbf{r}}^2 = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2),$$

the potential energy

$$V = mgy = mgq_2$$

and thus our Lagrangian is

$$L = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2) - mgq_2.$$

The equations of motion are
x-direction

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$
$$\Leftrightarrow$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_1) = 0$$
$$\Leftrightarrow$$
$$m\ddot{q}_1 = m\ddot{x} = 0$$
$$\Leftrightarrow$$
$$x(t) = v_0 \cos\alpha t$$

y-direction

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0 \\
&\Leftrightarrow \\
&\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}_2) + mg = 0 \\
&\Leftrightarrow \\
&m\ddot{q}_2 = m\ddot{y} = -mg \\
&\Leftrightarrow \\
&y(t) = v_0 \sin\alpha t - \frac{1}{2}gt^2.
\end{aligned}
$$

So our solution is

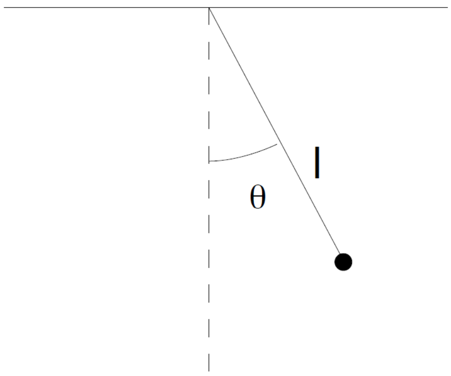
$$
\begin{aligned}
x(t) &= v_0 \cos\alpha t \\
y(t) &= v_0 \sin\alpha t - \frac{1}{2}gt^2.
\end{aligned}
$$

b)
As said in the lecture notes, we cannot intoduce friction into the Lagrangian. The Lagrange equation is valid only for conservative systems but let's look a few candidates for the friction term and prove that they do not work. We can separate our Lagrangian as $L = L_0 + L_i$, where the possible candidates are $L_1 = q^2, L_2 = \dot{q}^2$ and $L_3 = q\dot{q}$. The term L_0 represents the Lagrangian without friction. Because of the linearity in the Lagrange equation we can always multiply our terms L_i with a constant and we only need to calculate

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_1}{\partial \dot{q}}\right) - \frac{\partial L_1}{\partial q} &= -2q \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_2}{\partial \dot{q}}\right) - \frac{\partial L_2}{\partial q} &= 2\ddot{q} \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_3}{\partial \dot{q}}\right) - \frac{\partial L_3}{\partial q} &= 0.
\end{aligned}
$$

So no trial function gives us any terms just depending on the velocity $\dot{q}$ and thus they do no produce friction to the equation of motion.

3. **Solution**:



The kinetic energy

$$T = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2, \qquad (r = l \Rightarrow \dot{r} = 0)$$

the potential energy

$$V = -mgy = -mgl\cos\theta$$

and the Lagrangian is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta.$$

The equation of motion is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0\\
&\Leftrightarrow\\
&ml^2\ddot{\theta} + mgl\sin\theta = 0\\
&\Leftrightarrow\\
&\ddot{\theta} + \frac{g}{l}\sin\theta = 0.
\end{aligned}$$

If we study small oscillations meaning $\theta << 1$, then it holds that

$$\sin\theta = \theta - \frac{1}{6}\theta^3 + \cdots \approx \theta$$

but the angle must be in radians. Now our equation of motion simplifies as

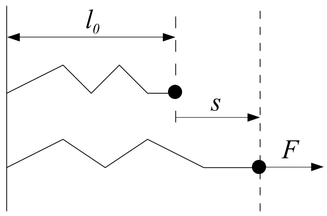
$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

which is the equation of the harmonic oscillator. Thus the solution is

$$\theta(t) = A\cos(\sqrt{\frac{g}{l}}t + \delta)$$

where A (describing the amplitude of oscillations) and δ (a phase constant) are determinated by the initial conditions.

4. **Solution**:



(a) The force needed to stretch a spring by length s from its equilibrium length is

$$F = ks$$

where k is the spring constant. In order to determine the potential energy of the spring, we calculate the work that is made when s is changed from 0 to s_1 (the subindex 1 is used to distinguish the instantaneous stretching from the final stretching),

$$\begin{aligned} W &= \int_0^{s_1} F\mathrm{d}s \\ &= \int_0^{s_1} ks\mathrm{d}s \\ &= \frac{1}{2}k \Big/_0^{s_1} s^2 \\ &= \frac{1}{2}ks_1^2. \end{aligned}$$

This gives the potential energy $V_s = \frac{1}{2}ks^2$, where s is the strecthing of the spring. Taking s negative (compressed spring) does not change anything in the formulas above.

(b) A good choice for the generalized coordinate is the vertical coordinate y measured from the lower end from the unstretched spring. The kinetic energy is

$$T = \frac{1}{2}m\dot{y}^2,$$

and the potential energy is (as a constant term can be neglected)

$$V = V_s + V_g = \underbrace{\frac{1}{2}ky^2}_{=\text{the spring}} + \underbrace{mgy}_{=\text{gravity}}$$

thus the Lagrangian is

$$L = \frac{1}{2}m\dot{y}^2 - \frac{1}{2}ky^2 - mgy.$$

So the equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0$$
$$\Leftrightarrow$$
$$m\ddot{y} + ky + mg = 0$$
$$\Leftrightarrow$$
$$\ddot{y} + \frac{k}{m}y + g = 0$$
$$\Leftrightarrow$$
$$\ddot{y} + \omega^2 y = -g,$$

where we have simplified the equation of motion by using a notation

$$\omega^2 = \frac{k}{m}.$$

The new equation of motion is a perfect second-order linear constant coefficient ordinary differential equation. The general solution can be found with the following procedure. The first step is to find the general solution of the homogeneous equation (dropping $-g$ on the right hand side):

$$\ddot{y}_h + \omega^2 y_h = 0 \Rightarrow y_h = A\cos(\omega t + \delta)$$

The second step is to find a special solution y_0 of the full equation. The trial function for the special solution has to be the same order as the term in the right-hand side in our original equation. In this case that term is constant, so our trial is also a constant y_0. This constant is

$$\omega^2 y_0 = -g \Rightarrow y_0 = -\frac{g}{\omega^2} = -\frac{mg}{k}.$$

Now the general solution is the sum of the two separate solutions

$$y = y_h + y_0 = A\cos(\omega t + \delta) - \frac{mg}{k}.$$

5. **Solution**:
a)

Method 1

(in Cartesian system)
The coordinate vector in the spherical system is

$$\mathbf{r} = r\cos\phi\sin\theta\mathbf{i} + r\sin\phi\sin\theta\mathbf{j} + r\cos\theta\mathbf{k}$$

and so the velocity is

$$\begin{aligned}\mathbf{v} &= [\dot{r}\cos\phi\sin\theta - r\dot{\phi}\sin\phi\sin\theta + r\dot{\theta}\cos\phi\cos\theta]\mathbf{i} \\ &+ [\dot{r}\sin\phi\sin\theta + r\dot{\phi}\cos\phi\sin\theta + r\dot{\theta}\sin\phi\cos\theta]\mathbf{j} \\ &+ [\dot{r}\cos\theta - r\dot{\theta}\sin\theta]\mathbf{z}.\end{aligned}$$

Now our velocity has the form $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$. The kinetic energy is

$$T = \frac{1}{2}m(v_x^2 + v_y^2 + v_z^2),$$

that has components

$$\begin{aligned}v_x^2 &= \dot{r}^2\cos^2\phi\sin^2\theta + r^2\dot{\phi}^2\sin^2\phi\sin^2\theta + r^2\dot{\theta}^2\cos^2\phi\cos^2\theta \\ &+ 2(-\dot{r}r\dot{\phi}\cos\phi\sin\phi\sin^2\theta + \dot{r}r\dot{\theta}\cos^2\phi\sin\theta\cos\theta \\ &- r^2\dot{\phi}\dot{\theta}\sin\phi\cos\phi\sin\theta\cos\theta)\end{aligned}$$

$$\begin{aligned}v_y^2 &= \dot{r}^2\sin^2\phi\sin^2\theta + r^2\dot{\phi}^2\cos^2\phi\sin^2\theta + r^2\dot{\theta}^2\sin^2\phi\cos^2\theta \\ &+ 2(-\dot{r}r\dot{\phi}\cos\phi\sin\phi\sin^2\theta + \dot{r}r\dot{\theta}\sin^2\phi\sin\theta\cos\theta \\ &- r^2\dot{\phi}\dot{\theta}\sin\phi\cos\phi\sin\theta\cos\theta)\end{aligned}$$

$$v_z^2 = \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\dot{\theta}^2\sin^2\theta.$$

Noting that some terms cancel each other and using the relation $\sin^2\phi + \cos^2\phi = 1$ we get a result:

$$v_x^2 + v_y^2 + v_z^2 = \dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2$$

$$\Rightarrow T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2).$$

Method 2

(in spherical system)
This proof is based on the fact

$$\mathrm{d}\hat{r} = \sin\theta\mathrm{d}\phi\hat{\phi} + \mathrm{d}\theta\hat{\theta}$$

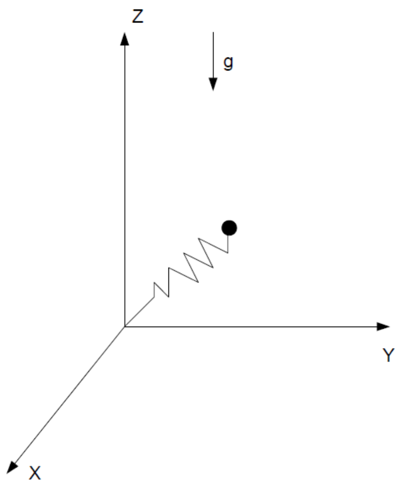
which is illustrated in the last page figures. Now

$$\begin{aligned}
\mathbf{r} &= r\hat{r} \\
\Rightarrow \mathbf{v} &= \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \\
&= \dot{r}\hat{r} + r\frac{\mathrm{d}\hat{r}}{\mathrm{d}t} \\
&= \dot{r}\hat{r} + r\frac{\mathrm{d}\theta}{\mathrm{d}t}\hat{\theta} + r\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}t}\hat{\phi} \\
&= \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\phi}\hat{\phi} \\
\Rightarrow T &= \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2)
\end{aligned}$$

where we used the facts

$$\hat{r}\cdot\hat{r} = \hat{\phi}\cdot\hat{\phi} = \hat{\theta}\cdot\hat{\theta} = 1$$
$$\hat{r}\cdot\hat{\phi} = \hat{r}\cdot\hat{\theta} = \hat{\phi}\cdot\hat{\theta} = 0.$$

b)



The kinetic energy is the same as in a). The spring only depends on the deviation from equilibrium, so the spring produces a potential

$$V_1(r) = \frac{1}{2}k(r - r_0)^2$$

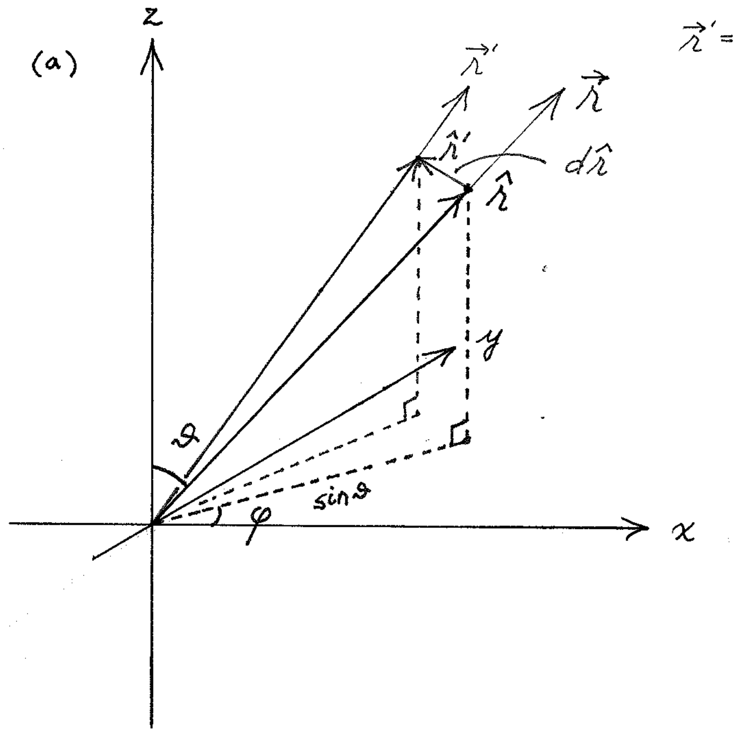
where r_0 is the rest lenght of the spring. Furthermore gravity gives us also a potential. Let's choose that gravity is parallel to the z-direction. Then the gravitional potential is

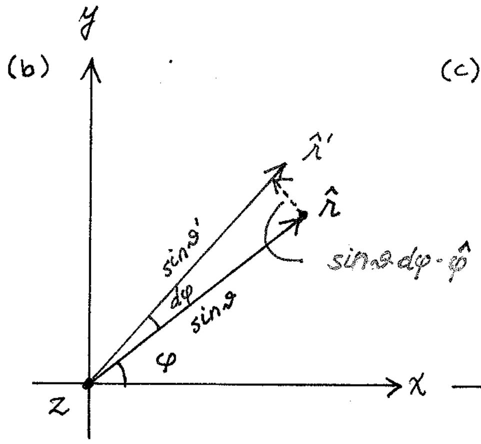
$$V_2(z) = mgz = mgr\cos\theta.$$

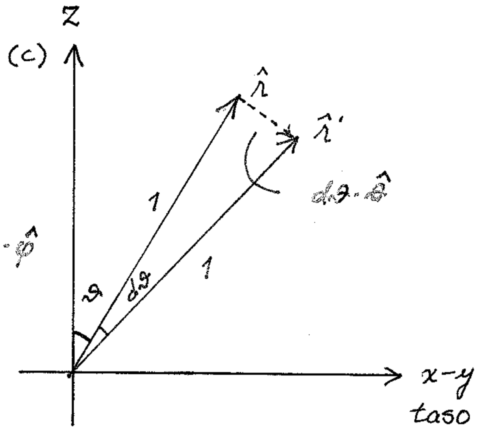
Thus the Lagrangian is

$$\begin{aligned}
L &= T - (V_1 + V_2) \\
&= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2\sin^2\theta + r^2\dot{\theta}^2) - \frac{1}{2}k(r - r_0)^2 - mgr\cos\theta.
\end{aligned}$$

$$\vec{r}\,' = \vec{r} + \frac{d\vec{r}}{dt}\cdot dt$$







Kuvat (b), (c) $\Rightarrow$ $d\hat{r} = \sin\vartheta\, d\varphi\cdot\hat{\varphi} + d\vartheta\cdot\hat{\vartheta}$