1. **Solution**:
The Lagrangian for the whole system is the sum of the Lagrangians $L = L_1 + L_2$: mass point 1 (mass m_1) and mass point 2 (mass m_2).

particle 1

Particle 1 (the attachment point) has the location

$$\mathbf{r}_1 = x\mathbf{i}$$

and thus the kinetic energy

$$T_1 = \frac{1}{2}m_1\dot{x}.$$

We can choose that the particle 1 has potential $V_1 = 0$.

particle 2

Particle 2 (the pendulum) has the location

$$\mathbf{r}_1 = (x + l\sin\phi)\mathbf{i} - l\cos\phi\mathbf{j}$$

and the velocity

$$\mathbf{v}_2 = \dot{\mathbf{r}}_2 = (\dot{x} + l\cos\phi\dot{\phi})\mathbf{i} + l\sin\phi\dot{\phi}\mathbf{j}.$$

Thus the kinetic energy has the form

$$\begin{aligned}
T_2 &= \frac{1}{2}m_2v_2^2 \\
&= \frac{1}{2}m_2\mathbf{v}_2\cdot\mathbf{v}_2 \\
&= \frac{1}{2}m_2[(\dot{x} + l\cos\phi\dot{\phi})^2 + l^2\sin\phi^2\dot{\phi}^2] \\
&= \frac{1}{2}m_2[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\cos^2\phi\dot{\phi}^2 + l^2\sin^2\phi\dot{\phi}^2] \\
&= \frac{1}{2}m_2[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2].
\end{aligned}$$

When we calculate the potential energy we have to remember our earlier choice. Then the potential is

$$V_2 = -m_2gl\cos\phi.$$

Now the Lagrangian for the whole system is

$$\begin{aligned}
L &= L_1 + L_2 \\
&= T_1 + T_2 - (V_1 + V_2) \\
&= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2l\dot{x}\dot{\phi}\cos\phi + \frac{1}{2}m_2l^2\dot{\phi}^2 + m_2gl\cos\phi.
\end{aligned}$$

2. **Solution**:
The particle has the position

$$\mathbf{r}_1 = [a\cos\omega t + l\sin\phi]\mathbf{i} + [a\sin\omega t - l\cos\phi]\mathbf{j}$$

and thus the velocity is

$$\mathbf{v}_1 = [-a\omega\sin\omega t + l\dot{\phi}\cos\phi]\mathbf{i} + [a\omega\cos\omega t + l\dot{\phi}\sin\phi]\mathbf{j}.$$

meaning that the kinetic energy is

$$\begin{aligned}
T &= \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} \\
&= \frac{1}{2}m[(-a\omega\sin\omega t + l\dot{\phi}\cos\phi)^2 + (a\omega\cos\omega t + l\dot{\phi}\sin\phi)^2] \\
&= \frac{1}{2}m[a^2\omega^2\sin^2\omega t - 2al\omega\dot{\phi}\cos\phi\sin\omega t + l^2\dot{\phi}^2\cos^2\phi \\
&+ a^2\omega^2\cos^2\omega t + 2al\omega\dot{\phi}\sin\phi\cos\omega t + l^2\dot{\phi}^2\sin^2\phi] \\
&= \frac{1}{2}m[a^2\omega^2\underbrace{(\sin^2\omega t + \cos^2\omega t)}_{=1} + 2al\omega\dot{\phi}\underbrace{(\sin\phi\cos\omega t - \cos\phi\sin\omega t)}_{=\sin(\phi-\omega t)} + l^2\dot{\phi}^2\underbrace{(\sin^2\omega t + \cos^2\omega t)}_{=1}] \\
&= \frac{1}{2}ma^2\omega^2 + mla\omega\dot{\phi}\sin(\phi-\omega t) + \frac{1}{2}ml^2\dot{\phi}^2
\end{aligned}$$

The potential energy is

$$\begin{aligned}
V &= mgy, \qquad y = a\sin\omega t - l\cos\phi \\
&= mg(a\sin\omega t - l\cos\phi) \\
&= mga\sin\omega t - mgl\cos\phi.
\end{aligned}$$

Thus the Lagrangian is

$$\begin{aligned}
L &= T - V \\
&= \frac{1}{2}ml^2\dot{\phi}^2 + mla\omega\dot{\phi}\sin(\phi-\omega t) + mgl\cos\phi - mga\sin\omega t + \frac{1}{2}ma^2\omega^2.
\end{aligned}$$

Let's denote

$$L_0 = -mga\sin\omega t + \frac{1}{2}ma^2\omega^2.$$

Now we see that

$$\frac{\partial L_0}{\partial \dot{q}} = \frac{\partial L_0}{\partial q} = 0,$$

where in our case $q = \phi$. So L_0 does not contribute to the equation of motion.

3. **Solution**:
Before proving anything, let's recall some useful results:

$$\nabla \times \nabla\phi \equiv 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) \equiv 0.$$

These results hold for an arbitrary scalar field ϕ and vector field $\mathbf{A}$. Of course, we assume that the needed derivates exist (this is usually the case in physics). If you do not believe, you can prove the results by simple calculations.

a)

Now we have

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Thus

$$\nabla \cdot \mathbf{B} = \nabla \cdot \nabla \times \mathbf{A} = 0$$

and

$$\begin{aligned}
\nabla \times \mathbf{E} &= \nabla \times (-\nabla\phi - \partial_t \mathbf{A}) \\
&= -\underbrace{\nabla \times \nabla\phi}_{=0} - \underbrace{\nabla \times \partial_t}_{=\partial_t \nabla\times} \mathbf{A} \\
&= -\partial_t \underbrace{\nabla \times \mathbf{A}}_{=\mathbf{B}} \\
&= -\partial_t \mathbf{B}.
\end{aligned}$$

So the fields $\mathbf{E}$ and $\mathbf{B}$ defined by the scalar field ϕ and the vector field $\mathbf{A}$ produce two of the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

b)

Now we make a gauge transformation to the potentials i. e.

$$\phi' = \phi - \partial_t \chi$$

$$\mathbf{A}' = \mathbf{A} + \nabla\chi$$

that defines new fields $\mathbf{E}' = -\nabla\phi' - \partial_t \mathbf{A}'$ and $\mathbf{B}' = \nabla \times \mathbf{A}'$. By simple calcutations we get

$$\begin{aligned}
\mathbf{E}' &= -\nabla\phi' - \partial_t \mathbf{A}' \\
&= -\nabla(\phi - \partial_t \chi) - \partial_t(\mathbf{A} + \nabla\chi) \\
&= -\nabla\phi + \partial_t \nabla\chi - \partial_t \mathbf{A} - \partial_t \nabla\chi, \qquad (\partial_t \nabla = \nabla \partial_t) \\
&= -\nabla\phi - \partial_t \mathbf{A} \\
&= \mathbf{E}
\end{aligned}$$

and

$$\begin{aligned}\mathbf{B}' &= \nabla \times \mathbf{A}' \\ &= \nabla \times (\mathbf{A} + \nabla\chi) \\ &= \underbrace{\nabla \times \mathbf{A}}_{=\mathbf{B}} + \underbrace{\nabla \times \nabla\chi}_{=0} \\ &= \mathbf{B}.\end{aligned}$$

So we see that the gauge transformation does not change the fields. We call that the fields are *gauge invariants*.

4. **Solution**:
We have a charged particle (the mass m and the charge q) in a magnetic field $\mathbf{B} = B\mathbf{k}$. Let's choose a potential as $\mathbf{A} = -By\mathbf{i}$. Now

$$\nabla \times \mathbf{A} = \nabla \times (-By)\mathbf{i} = B\mathbf{k} = \mathbf{B}.$$

So our potential gives the correct magnetic field.
The velocity of the particle is $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$. The potential energy using the result from lectures is

$$\begin{aligned}V &= q(\phi - \mathbf{v} \cdot \mathbf{A}) \\ &= -q(\dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}) \cdot (-By)\mathbf{i} \\ &= qB\dot{x}y\end{aligned}$$

Note that one can always choose the scalar potential to be zero $\phi \equiv 0$ (because of the result of previous question 3b). Now the kinetic energy

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(\mathbf{v} \cdot \mathbf{v}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

Thus the Lagrangian of the particle is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - qB\dot{x}y.$$

The equation of motion for the particle is given by the Lagrange equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0.$$

x-direction

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Leftrightarrow m\ddot{x} - qB\dot{y} = 0$$

y-direction

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Leftrightarrow m\ddot{y} + qB\dot{x} = 0$$

z-direction

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Leftrightarrow m\ddot{z} = 0$$

The equation of motion in z-direction is easy to solve:

$$\ddot{z} = 0 \Rightarrow z(t) = z(0) + \dot{z}(0)t.$$

This result means that the particle in the z-direction is moving with a constant velocity. In the xy-plane we have to solve a set of equations:

$$\begin{aligned}\ddot{x} &= \frac{qB}{m}\dot{y} \Rightarrow \ddot{x} = -\omega\dot{y}\\ \ddot{y} &= -\frac{qB}{m}\dot{x} \Rightarrow \ddot{y} = \omega\dot{x}\end{aligned}$$

where we use a notation

$$\omega = -\frac{qB}{m}.$$

The equations can be solved by many ways. One method is

$$\begin{aligned}&\ddot{x} = -\omega\dot{y}\\ &\Leftrightarrow\\ &\frac{\mathrm{d}}{\mathrm{d}t}\ddot{x} = -\omega\frac{\mathrm{d}}{\mathrm{d}t}\dot{y}, \qquad \ddot{y} = \omega\dot{x}\\ &\Leftrightarrow\\ &\dddot{x} = -\omega^2\dot{x}, \qquad \text{notation}\, v_x = \dot{x}\\ &\Leftrightarrow\\ &\ddot{v}_x + \omega^2 v_x = 0, \qquad \text{harmonic oscillator}\\ &\Leftrightarrow\\ &v_x = A\sin(\omega t + \phi_0)\\ &x = \int v_x \mathrm{d}t = -\frac{A}{\omega}\cos(\omega t + \phi_0) + x_0\end{aligned}$$

and thus

$$\begin{aligned}&\ddot{x} = -\omega\dot{y}\\ &\Leftrightarrow\\ &\dot{y} = -\frac{1}{\omega}\ddot{x} = -A\cos(\omega t + \phi_0)\\ &\Leftrightarrow\\ &y = \int \dot{y}\mathrm{d}t = -\frac{A}{\omega}\sin(\omega t + \phi_0) + y_0.\end{aligned}$$

Now let's denote

$$r_0 \equiv -\frac{A}{\omega}.$$

Thus our solution is

$$x = r_0 \cos(\omega t + \phi_0) + x_0$$

$$y = r_0 \sin(\omega t + \phi_0) + y_0.$$

where these six free parameters r_0, ϕ_0, x_0, y_0, $z(0)$ and $\dot{z}(0)$ are determined by the initial values.