1. **Solution**:

a) The kinetic energy is

$$T = \frac{1}{2}m\dot{x}^2$$

and the potential

$$V = k6x(x-2)$$

so the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - 6kx(x-2).$$

The equation of motion is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \\
&\Leftrightarrow \\
&\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) + 6k(x-2) + 6kx = 0 \\
&\Leftrightarrow \\
&m\ddot{x} + 12kx = 12k
\end{aligned}$$

The solution is $x = x_h + x_0$, where x_h is the solution for the homogeneous equation and x_0 is the solution for the special case. The homogeneous solution is gotten by solving the homogeneous equation:

$$m\ddot{x}_h + 12kx = 0.$$

The above equation is the equation of the harmonic oscillator and thus the solution is

$$x_h = a\cos(\sqrt{12}\omega t + \delta), \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}.$$

The constant a and δ depend on the initial values. For the special solution, we make a trial function $x_0 = b = \text{constant}$. Now we put this trial into our original equation and solve the constant b:

$$\begin{aligned}
&m\ddot{x}_0 + 12kx_0 = 12k \\
&\Leftrightarrow \\
&12kb = 12k \\
&\Leftrightarrow \\
&b = 1.
\end{aligned}$$

The general solution for the equation of motion is the linear combination of the homogeneous and special solution (the superpositon principle):

$$x(t) = x_h + x_0 = a\cos(\sqrt{12}\omega t + \delta) + 1.$$

b)
The kinetic energy is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

and the potential

$$V = \frac{1}{2}(k_1 x^2 + k_2 y^2 + k_3 z^2)$$

so the Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}(k_1 x^2 + k_2 y^2 + k_3 z^2).$$

The equations of motion are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} &= 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} m\ddot{x} + k_1 x &= 0 \\ m\ddot{y} + k_2 y &= 0 \\ m\ddot{z} + k_3 z &= 0 \end{aligned}$$

These equations are again equations of the harmonic oscillators and thus solutions are already known:

$$\begin{aligned} x &= a_1\cos(\omega_1 t + \delta_1) \\ y &= a_2\cos(\omega_2 t + \delta_2) \\ z &= a_3\cos(\omega_3 t + \delta_3) \end{aligned}$$

where

$$\omega_i = \sqrt{\frac{k_i}{m}} \qquad i = x, y, z.$$

In the vector notation the solution is

$$\mathbf{r} = a_1\cos(\omega_1 t + \delta_1)\mathbf{i} + a_2\cos(\omega_2 t + \delta_2)\mathbf{j} + a_3\cos(\omega_3 t + \delta_3)\mathbf{k}.$$

2. **Solution**:
On the surface of the cylinder with radius R the location of the particle is given as

$$\mathbf{r} = R\cos\phi\mathbf{i} + R\sin\phi\mathbf{j} + z\mathbf{k}.$$

Now the particle has velocity

$$\mathbf{v} = -R\dot{\phi}\sin\phi\mathbf{i} + R\dot{\phi}\cos\phi\mathbf{j} + \dot{z}\mathbf{k}$$

and thus it has the kinetic energy

$$\begin{aligned} T &= \frac{1}{2}mv^2 \\ &= \frac{1}{2}m(R^2\dot{\phi}^2\sin^2\phi + R^2\dot{\phi}^2\cos^2\phi + \dot{z}^2) \\ &= \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2). \end{aligned}$$

The potential V associated to the force $\mathbf{F} = -k\mathbf{r}$ is

$$\begin{aligned} V &= \frac{1}{2}kr^2 \\ &= \frac{1}{2}k(\mathbf{r}\cdot\mathbf{r}) \\ &= \frac{1}{2}k(R^2\cos^2\phi + R^2\sin^2\phi + z^2) \\ &= \frac{1}{2}k(R^2 + z^2) \end{aligned}$$

and you can check that now holds $\mathbf{F} = -\nabla V$. Of course, you could calculate the potential from the formula $\mathbf{F} = -\nabla V$. After this we can write the Lagrangian

$$L = T - V = \frac{1}{2}m(R^2\dot{\phi}^2 + \dot{z}^2) - \frac{1}{2}k(R^2 + z^2).$$

and the the equations of motion are

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{z}}\right) - \frac{\partial L}{\partial z} &= 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(mR^2\dot{\phi}) &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{z}) + kz &= 0 \end{aligned}$$

$$\Leftrightarrow$$

$$\ddot{\phi} = 0$$

$$\ddot{z} + \frac{k}{m} z = 0$$

The solution in the z-direction is clearly again

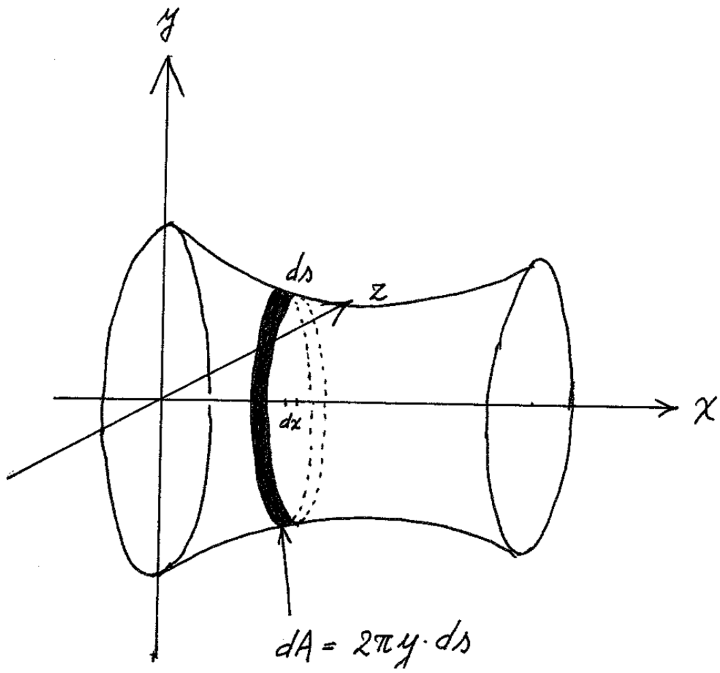
$$z(t) = A\cos(\omega t + \delta), \qquad \omega = \sqrt{\frac{k}{m}}$$

and the angular equation is easy to solve:

$$\phi(t) = \omega_0 t + \phi_0.$$

The terms a, δ, ω_0 and ϕ_0 are constants determined by the initial values.

3. **Solution**:



The line element is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \sqrt{1 + \frac{\mathrm{d}x^2}{\mathrm{d}y^2}}\mathrm{d}x = \sqrt{1 + \dot{y}^2}\mathrm{d}x.$$

Thus the surface element is

$$\mathrm{d}A = 2\pi y\sqrt{1 + \dot{y}^2}\mathrm{d}x.$$

and the corresponding area is determined by integrating the surface element

$$A = \int_{x_1}^{x_2} \mathrm{d}A = 2\pi \int_{x_1}^{x_2} y\sqrt{1+\dot{y}^2}\mathrm{d}x.$$

Like in the lectures we are looking for an extreme for the quantity

$$\int_{x_1}^{x_2} f(y,\dot{y},x)\mathrm{d}x$$

where

$$f(y,\dot{y},x) = y\sqrt{1+\dot{y}^2}.$$

Because f does not depend on the value x ($f(y,\dot{y},x) \to f(y,\dot{y})$), the formula in the lectures holds

$$f - \dot{y}\frac{\partial f}{\partial \dot{y}} = C = \text{constant}.$$

Now we insert our function f into the above formula:

$$y\sqrt{1+\dot{y}^2} - \dot{y}\frac{\partial}{\partial \dot{y}}(y\sqrt{1+\dot{y}^2}) = C$$

$$\Leftrightarrow$$

$$y\sqrt{1+\dot{y}^2} - \frac{y\dot{y}^2}{\sqrt{1+\dot{y}^2}} = C$$

$$\Leftrightarrow$$

$$\frac{y(1+\dot{y}^2) - y\dot{y}^2}{\sqrt{1+\dot{y}^2}} = C$$

$$\Leftrightarrow$$

$$\frac{y}{\sqrt{1+\dot{y}^2}} = C$$

Now we show that

$$y(x) = a\cosh\left(\frac{x-b}{a}\right)$$

is the solution for the equation:

$$C = \frac{y}{\sqrt{1+\dot{y}^2}} = \frac{a\cosh\left(\frac{x-b}{a}\right)}{\sqrt{1+\sinh^2\left(\frac{x-b}{a}\right)}} = \frac{a\cosh\left(\frac{x-b}{a}\right)}{\cosh\left(\frac{x-b}{a}\right)} = a\,.$$

The general solution can be acquired by solving the above differential equation:

$$y = C\sqrt{1+\dot{y}^2}$$

$\Leftrightarrow$

$$y^2 = C(1+\dot{y}^2) \qquad \text{let's solve}\,\dot{y}$$

$\Leftrightarrow$

$$\dot{y}^2 = \frac{y^2}{C^2} - 1$$

$\Leftrightarrow$

$$\dot{y} = \pm\sqrt{\frac{y^2}{C^2} - 1} \qquad \text{it is a separable equation}\,\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$\Leftrightarrow$

$$\mathrm{d}x = \left(\sqrt{\frac{y^2}{C^2} - 1}\right)^{-1}\mathrm{d}y \qquad \text{let's integrate}$$

$\Leftrightarrow$

$$\int \mathrm{d}x = \int \left(\sqrt{\frac{y^2}{C^2} - 1}\right)^{-1}\mathrm{d}y \qquad \text{make the change of the variable}\,y = C\mu \Rightarrow \mathrm{d}y = C\mathrm{d}\mu$$

$\Leftrightarrow$

$$x = C\int \frac{\mathrm{d}\mu}{\sqrt{\mu^2 - 1}} \qquad \int \frac{\mathrm{d}\mu}{\sqrt{\mu^2-1}} = \mathrm{arccosh}\mu$$

$\Leftrightarrow$

$$x = C\mathrm{arccosh}\frac{y(x)}{C} + D \qquad \text{let's solve}\,y$$

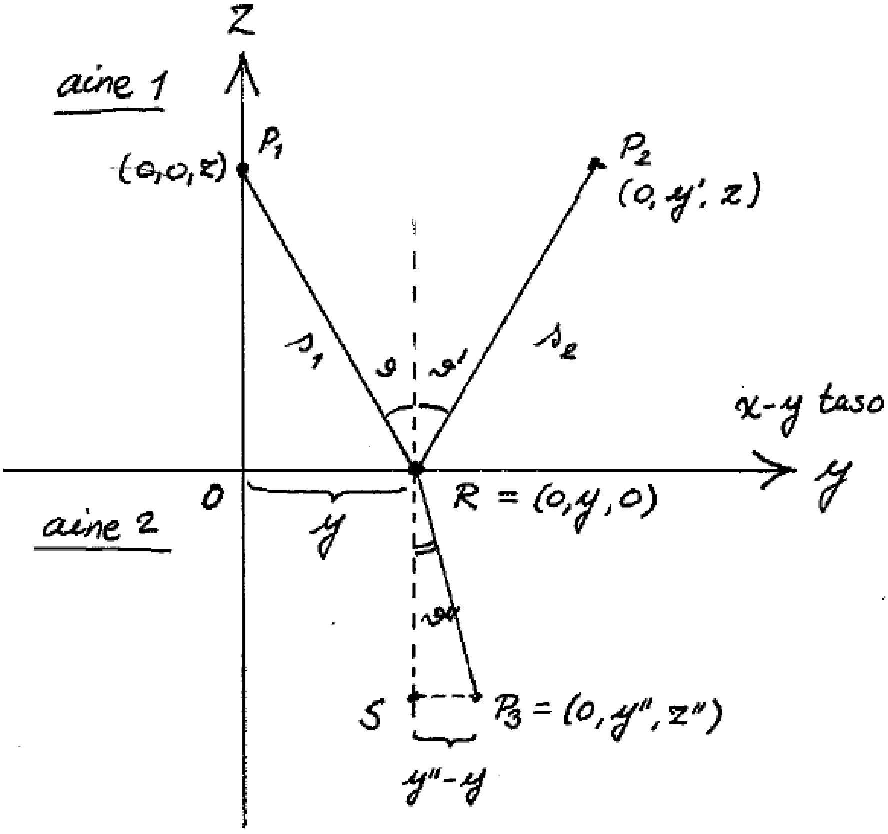
$\Leftrightarrow$

$$\mathrm{arccosh}\frac{y(x)}{C} = \frac{x-D}{C}$$

$\Leftrightarrow$

$$y(x) = C\cosh(\frac{x-D}{C}).$$

4. **Solution**:



a)

Let's denote the path of the particle (ray) with s. Because the velociy is constant in homogeneous matter, time spended by traveling the path s is

$$T = \frac{s(T)}{c}.$$

Thus the minimization of the treduce to the minimization of the path. In the lectures this minimization was done in the xy-plane and the result was a straight line. But let's do this minimization of the particle's path in the three dimension space:

$$s(T) = \int_0^T \mathrm{d}s(t) = \int_0^T \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}t = \int_0^T \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\mathrm{d}t,$$

where we use results

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}$$

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}.$$

The integrand of the minimization is

$$f = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2} = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}$$

and using the last notation on the above our three Euler equations (x, y, z) will become only one

$$\frac{\partial f}{\partial \mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\mathbf{r}}}\right) = 0.$$

In the above formula we use a notation

$$\frac{\partial f}{\partial \mathbf{r}} \equiv \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k},$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Now we do the minimization with our Euler equation

$$\begin{aligned}
&\frac{\partial f}{\partial \mathbf{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{\mathbf{r}}}\right) = 0 \\
&0 - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\frac{1}{\sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}}}2\dot{\mathbf{r}}\right) = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\frac{\dot{\mathbf{r}}}{C} = 0, \quad C = \sqrt{\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}} = \text{constant} \\
&\ddot{\mathbf{r}} = 0
\end{aligned}$$

that implies $\mathbf{r} =$ straight line.

b)

the reflection law

We minimize time spend between points P_1 and P_2:

$$T_1 = \frac{s_1 + s_2}{c_1} = \frac{\sqrt{y^2 + z^2}}{c_1} + \frac{\sqrt{(y' - y)^2 + z^2}}{c_1}.$$

We can think that points P_1 and P_2 are fixed and then the only point that can move is the reflection point. Thus our unknown variable is y and we minimize respect to that. The minimalization requirement is

$$\begin{aligned}
&\frac{\mathrm{d}T_1}{\mathrm{d}y} = 0 \\
&\frac{1}{c_1}\frac{y}{\sqrt{y^2 + z^2}} + \frac{1}{c_1}\frac{y - y'}{\sqrt{(y' - y)^2 + z^2}} = 0 \\
&\frac{y^2}{y^2 + z^2} = \frac{(y - y')^2}{(y - y')^2 + z^2} \\
&y^2\left[(y - y')^2 + z^2\right] = (y^2 + z^2)(y - y')^2 \\
&y^2 z^2 = z^2 (y - y')^2 \\
&y = -y + y' \\
&y = \frac{1}{2}y'
\end{aligned}$$

which means that the reflection point is exactly in the middle of P_1 and P_2. This implies that the income angle θ and the refection angle θ' are same

<u>*the Snell refract law*</u>

This time we minimize time but from the point P_1 to the point P_3:

$$T_2 = \frac{s_1}{c_1} + \frac{s_2}{c_2} = \frac{\sqrt{y^2+z^2}}{c_1} + \frac{\sqrt{(y''-y)^2+z''^2}}{c_2}$$

and thus

$$\begin{aligned}
&\frac{\mathrm{d}T_2}{\mathrm{d}y} = 0\\
&\frac{1}{c_1}\frac{y}{\sqrt{y^2+z^2}} + \frac{1}{c_2}\frac{y-y''}{\sqrt{(y-y'')^2+z''^2}} = 0\\
&\frac{1}{c_1}\frac{y}{s_1} + \frac{1}{c_2}\frac{y-y''}{s_2} = 0\\
&\frac{1}{c_1}\frac{y}{s_1} = \frac{1}{c_2}\frac{y''-y}{s_2}\\
&\frac{\sin\theta}{c_1} = \frac{\sin\theta''}{c_2}\\
&\frac{\sin\theta}{\sin\theta''} = \frac{c_1}{c_2}
\end{aligned}$$

Note: in the triangle $\mathrm{OP_1R}$ the opposite cathetus of the angle θ is OR that has lenght y and the lenght of the hypotenuse P_1 is s_1. Thus $\sin\theta = \frac{y}{s_1}$. Correspondly in the triangle $\mathrm{RP_3S}$ $\sin\theta'' = \frac{y''-y}{s_2}$.