1. **Solution**:

Let $L(q, \dot{q}, \ddot{q}, t)$. The equations of motion are determined by the Hamilton principle meaning that we have to minimize the action functional

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, \ddot{q}, t)\mathrm{d}t.$$

The minimization is done by the variation method:

$$\delta S = \int_{t_1}^{t_2} \delta L(q, \dot{q}, \ddot{q}, t)\mathrm{d}t = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta \dot{q} + \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q}\right]\mathrm{d}t$$

Now we get rid of the terms with time derivative of the variaton by partial integrating terms:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}}\delta \dot{q}\mathrm{d}t = \Big/_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}}\delta q - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\mathrm{d}t$$

$$\begin{aligned}\int_{t_1}^{t_2} \frac{\partial L}{\partial \ddot{q}}\delta \ddot{q}\mathrm{d}t &= \Big/_{t_1}^{t_2} \frac{\partial L}{\partial \ddot{q}}\delta \dot{q} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta \dot{q}\mathrm{d}t \\ &= \Big/_{t_1}^{t_2} \frac{\partial L}{\partial \ddot{q}}\delta \dot{q} - \Big/_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta q\mathrm{d}t + \int_{t_1}^{t_2} \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\delta q\mathrm{d}t.\end{aligned}$$

It is easy to see that good boundary conditions are that variation δq and its time derivate $\delta \dot{q}$ vanish in the starting and ending points. This led us to

$$\delta S = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right)\right]\delta q\mathrm{d}t.$$

The minimization of the action functional requires that $\delta S = 0$ for all δq. This holds if and only if

$$\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0.$$

If we assume that the Lagrangian is

$$L = -\frac{m}{2}q\ddot{q} - \frac{k}{2}q^2$$

then we get the equation of motion from the Lagrange equation:

$$\begin{aligned}&\frac{\partial L}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) = 0 \\ &-\frac{m}{2}\ddot{q} - kq + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(-\frac{m}{2}q\right) = 0 \\ &m\ddot{q} + kq = 0 \\ &\ddot{q} + \frac{k}{m}q = 0\end{aligned}$$

that is again the equation of the harmonic oscillator.

2. **Solution**:
Kinetic energy in polar coordinates is $T = (1/2)m(\dot{r}^2 + r^2\dot{\theta}^2)$ and potential energy is again $V = -mgr\cos\theta$. Thus the Lagrangian is

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta.$$

Note that the general coordinates for the system are truly r and θ. Earlier we assumed that r is not a general coordinate and got one variable problem. Now we assume that r is not constant and we will see how our differential constraint will fix the lenght of the pendulum. By differenting the lenght of the pendulum ($r(t) = l =$ constant) we get a differential constraint

$$\mathrm{d}r = 0.$$

Because we have parameters r and θ our constraint should have a form (look the lectures)

$$a_{1r}\mathrm{d}r + a_{1\theta}\mathrm{d}\theta = 0.$$

Now when we compare our constraint we get the coefficients

$$a_{1r} = 1 \quad \text{and} \quad a_{1\theta} = 0.$$

Using the Lagrange equation for the parameters r and θ with our constraint we get

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = a_{1r}\lambda_1 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = a_{1\theta}\lambda_1 \\
&\Leftrightarrow \\
&m\ddot{r} - mr\dot{\theta}^2 - mg\cos\theta = \lambda_1 \\
&mr^2\ddot{\theta} + mgr\sin\theta = 0
\end{aligned}$$

The Lagrange's multiplier is determined by our constraint. Because $\mathrm{d}r = 0$, it is that $\dot{r} = 0$ and $\ddot{r} = 0$. This means that

$$\lambda_1 = -mr\dot{\theta}^2 - mg\cos\theta.$$

The interpretation of this result is that λ_1 is the generalized force holding the length of the pendulum as constant i.e. the tension in the rope. The final note is that the equation for the angle is the familiar equation of the simple oscillator:

$$\ddot{\theta} + \frac{g}{r}\sin\theta = 0.$$

3. **Solution**:
The Lagrangian is

$$L = \frac{1}{2}m_1\dot{l_1}^2 + \frac{1}{2}m_2\dot{l_2}^2 + mgl_1 + mgl_2,$$

and a constraint is $l_1 + l_2 =$ constant. From the constraint we get a differential constraint

$$\mathrm{d}l_1 + \mathrm{d}l_2 = 0.$$

The Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{l_i}}\right) - \frac{\partial L}{\partial l_i} = a_{1i}\lambda_1$$

where $i = 1, 2$. We get the cofficients a_{li} from the differential constraint:

$$a_{11} = 1 \quad \text{and} \quad a_{12} = 1.$$

So Lagrange's equations are

$$\begin{aligned} m_1\ddot{l_1} - m_1 g &= \lambda_1 \\ m_2\ddot{l_2} - m_2 g &= \lambda_1. \end{aligned}$$

One can eliminate λ_1 by putting the equations to the same:

$$m_1\ddot{l_1} - m_1 g = m_2\ddot{l_2} - m_2 g.$$

From the differential constraint it follows that $\dot{l_1} = -\dot{l_2}$ and $\ddot{l_1} = -\ddot{l_2}$:

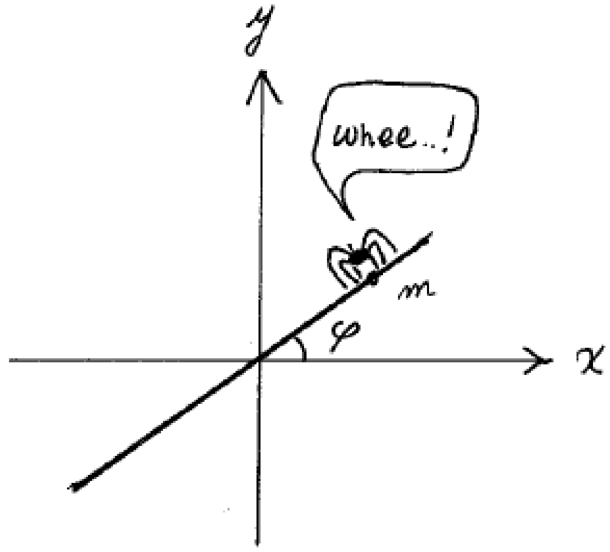
$$\begin{aligned} m_1\ddot{l_1} - m_1 g &= -m_2\ddot{l_1} - m_2 g \\ (m_1 + m_2)\ddot{l_1} &= (m_1 - m_2)g \\ \ddot{l_1} &= \frac{(m_1 - m_2)}{(m_1 + m_2)}g \\ \dot{l_1} &= \frac{(m_1 - m_2)}{(m_1 + m_2)}gt + \dot{l_1}(0) \\ l_1 &= \frac{1}{2}\frac{(m_1 - m_2)}{(m_1 + m_2)}gt^2 + \dot{l_1}(0)t + l_1(0) \end{aligned}$$

and in a similar way we can solve

$$l_2 = \frac{1}{2}\frac{(m_2 - m_1)}{(m_1 + m_2)}gt^2 + \dot{l_2}(0)t + l_2(0).$$

Now we can calculate the tension in the rope (we can substitute the acceleration $\ddot{l_1}$ or $\ddot{l_2}$):

$$\begin{aligned} \lambda_1 &= m_1\ddot{l_1} - m_1 g \\ &= m_1 g\left(\frac{m_1 - m_2}{m_1 + m_2} - 1\right) \\ &= -\frac{2m_1 m_2}{m_1 + m_2}g. \end{aligned}$$



4. **Solution**:
Spider's location is

$$\mathbf{r} = r\cos\phi\mathbf{i} + r\sin\phi\mathbf{j} \Rightarrow \dot{\mathbf{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

and kinetic energy

$$T_{\text{spider}} = \frac{1}{2}m\dot{\mathbf{r}}^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2).$$

The straw has kinetic energy

$$T_{\text{straw}} = \frac{1}{2}I\dot{\phi}^2.$$

Because the c.m. of the straw is located on the zero potential level, it does not have potential energy. The only potential energy is from the spider

$$V = mgy = mgr\sin\phi.$$

Now the Lagrangian is

$$L = T_{\text{spider}} + T_{\text{straw}} - V = \frac{1}{2}(I + mr^2)\dot{\phi}^2 + \frac{1}{2}m\dot{r}^2 - mgr\sin\phi$$

and so the Lagrange equation is

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0 \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left[(I + mr^2)\dot{\phi}\right] + mgr\cos\phi = 0 \\
&(I + mr^2)\ddot{\phi} + 2mr\dot{r}\dot{\phi} + mgr\cos\phi = 0
\end{aligned}$$

If we set the angular velocity $\dot{\phi} = \omega = \text{constant}$, we have $\phi = \omega t + \phi_0$ and $\ddot{\phi} = 0$. Using this assumption our equation of the motion reduces to

$$
\begin{aligned}
&2mr\dot{r}\omega + mgr\cos(\omega t + \phi_0) = 0 \\
&\dot{r} = -\frac{g}{2\omega}\cos(\omega t + \phi_0) = 0 \\
&r = -\frac{g}{2\omega^2}\sin(\omega t + \phi_0) + r_0
\end{aligned}
$$

It is clear that the kinetic energy of the straw does not affect our above solution (there is no I in the solution).