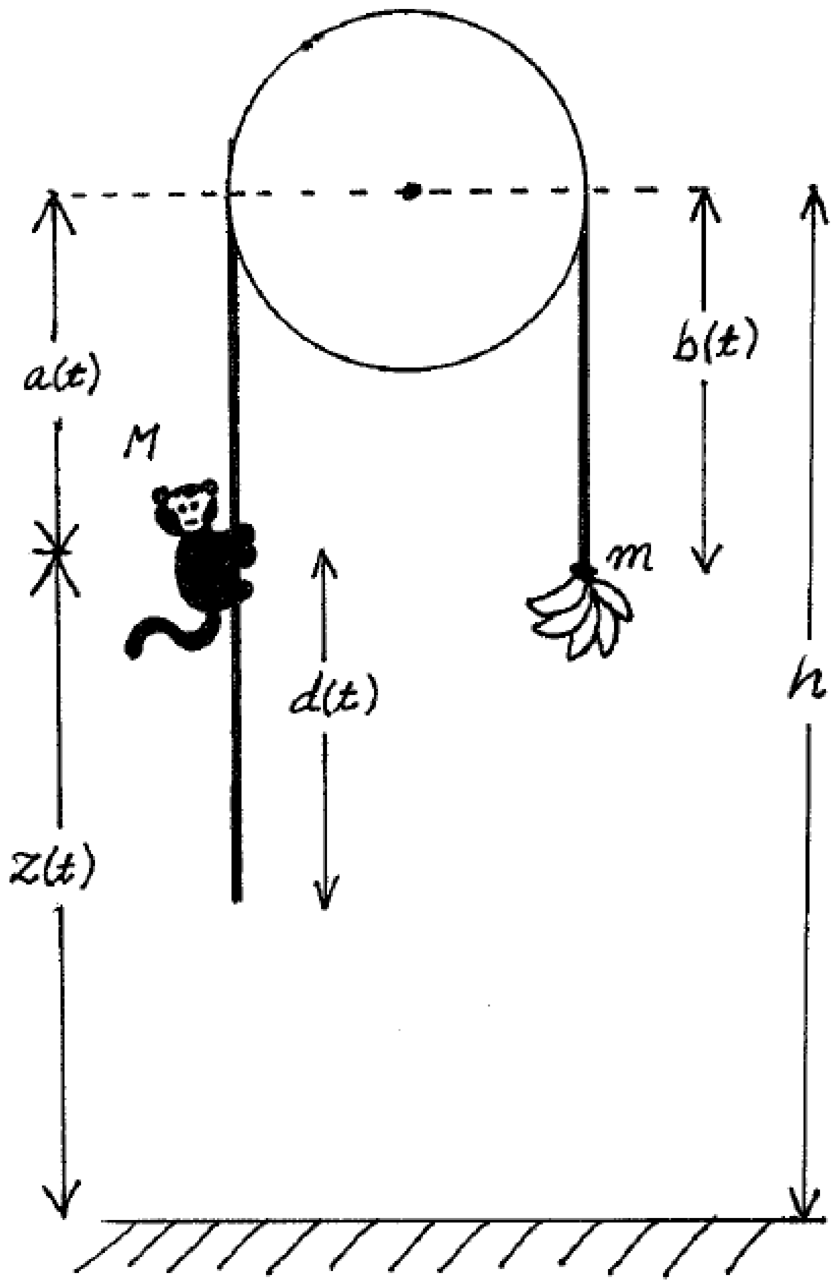
1. **Solution**:



Let us introduce two new auxiliary variables $a(t)$ and $b(t)$. Because the length of the rope is constant, we have in the system constraint

$$a(t) + b(t) + d(t) = \text{constant}. \tag{1}$$

the altitude of the monkey is $h - a(t)$ and that of the banana $h - b(t)$.

Now the kinetic energy of the system is

$$T = \frac{1}{2}m\dot{b}^2 + \frac{1}{2}M\dot{a}^2$$

and the total potential energy of the system is

$$V = mg(h - b) + Mg(h - a).$$

Thus the system has Lagrange function

$$L = T - V = \frac{1}{2}m\dot{b}^2 + \frac{1}{2}M\dot{a}^2 + mg(b - h) + Mg(a - h). \tag{2}$$

Now we have two ways to do the calculations.

with differential constraint

From equation 1 we get a differential constraint

$$\mathrm{d}a + \mathrm{d}b + \mathrm{d}d = 0. \tag{3}$$

The equations of motion are attained by the Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{l=1}^{m} \lambda_l a_{li}, \quad i = 1 \ldots m.$$

Because we have two variables $q_i = a, b$, then $n = 2$ and only one constraint meaning $m = 1$. The coefficients in the Lagrange equation are achieved by comparing the sum and the differential constraint 3. The coefficients are $a_{11} = 1$, $a_{12} = 1$ and $a_{1t} = \dot{d}(t)$(d is some known function of time, and $\mathrm{d}d(t) = \dot{d}(t)\mathrm{d}t$). Now we can make calculations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = a_{11}\lambda_1 = \lambda_1$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{b}}\right) - \frac{\partial L}{\partial b} = a_{12}\lambda_1 = \lambda_1$$

$$M\ddot{a} - Mg = \lambda_1$$
$$m\ddot{b} - mg = \lambda_1$$
$$M\ddot{a} - Mg = m\ddot{b} - mg, \tag{4}$$

where we eliminated the Lagrange multiplier λ_1. Because from the differential constraint

$$\mathrm{d}a + \mathrm{d}b + \mathrm{d}d = 0$$
$$\Rightarrow \dot{a} + \dot{b} + \dot{d} = 0$$
$$\Rightarrow \ddot{a} + \ddot{b} + \ddot{d} = 0$$
$$\Rightarrow \ddot{b} = -\ddot{a} - \ddot{d}$$

we get from equation 4

$$\begin{aligned} M\ddot{a} - Mg &= -m(\ddot{a} + \ddot{d}) - mg \\ (M+m)\ddot{a} + m\ddot{d} &= (M-m)g \end{aligned}$$

$$(M+m)\ddot{z} - m\ddot{d} = (m-M)g, \tag{5}$$

where we noticed $z = h - a \Rightarrow \ddot{a} = -\ddot{z}$.

with holonomic constraint

If we denote the length of the rope with l_0, we get from the equation 1 a holonomic constraint

$$a(t) + b(t) + d(t) = l_0 = \text{constant}. \tag{6}$$

We can use this constraint to solve e.g. $b(t)$:

$$b(t) = l_0 - a(t) - d(t).$$

Now we can substitute the solved b to Lagrange function (eq. 2)

$$L = L(a, \dot{a}) = \frac{1}{2}m(\dot{a} + \dot{d})^2 + \frac{1}{2}M\dot{a}^2 + mg(l_0 - a - d - h) + Mg(a - h)$$

and the Lagrange equation (note that we have now only one coordinate a)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{a}}\right) - \frac{\partial L}{\partial a} = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}t}\left(m(\dot{a} + \dot{d})^2 + M\dot{a}\right) + mg - Mg = 0 \\ &(m+M)\ddot{a} + m\ddot{d} + (m-M)g = 0, \quad z = h - a \\ &(M+m)\ddot{z} - m\ddot{d} = (m-M)g \end{aligned}$$

So we did get the same result. Let's solve the equation of the motion:

$$\begin{aligned} &(M+m)\ddot{z} - m\ddot{d} = (m-M)g \\ &\ddot{z} = \frac{m}{m+M}\ddot{d} + \frac{m-M}{m+M}g \\ &\dot{z}(t) - \underbrace{\dot{z}(0)}_{=0} = \frac{m}{m+M}(\dot{d}(t) - \underbrace{\dot{d}(0)}_{=0}) + \frac{m-M}{m+M}gt \\ &z(t) - z(0) = \frac{m}{m+M}d(t) + \frac{1}{2}\frac{m-M}{m+M}gt^2, \end{aligned}$$

where we used the initial values $\dot{d}(0) = \dot{z}(0) = 0$. If $M = m$, then it holds that $z(t) = \frac{1}{2}d(t)$ ($+z_0$ that is irrelevant for now), and thus the vertical distance is

$$\begin{aligned} z(t) - (h - b(t)) &= z(t) - h + l_0 - a(t) - d(t) \\ &= z(t) - a(t) - d(t) + (l_0 - h) \\ &= \frac{1}{2}d(t) - (h - \frac{1}{2}d(t)) - d(t) + (l_0 - h) \\ &= l_0 - 2h = \text{constant}. \end{aligned}$$

2. **Solution**:
The Lagrangian is familar

$$L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 gl\cos\phi.$$

It is easy to notice that the Lagrangian does not depent on the coordinate x. This means that the corresponding momentum p_x is conserved:

$$p_x = \frac{\partial L}{\partial \dot{x}} = (m_1 + m_2)\dot{x} + m_2 l\dot{\phi}\cos\phi.$$

and $\dot{p}_x = 0$ (check the Lagrange equation). Because there is explicit dependence on the coordinate ϕ, the momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + m_2 l\dot{x}\cos\phi$$

is not a constant of motion. The second important conserved quantity is the Hamiltonian H:

$$\begin{aligned} H &= \sum_i \dot{q}_i p_i - L \\ &= (m_1 + m_2)\dot{x}^2 + 2m_2 l\dot{x}\dot{\phi}\cos\phi + ml^2\dot{\phi}^2 \\ &- [\frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + \frac{1}{2}m_2 l^2\dot{\phi}^2 + m_2 gl\cos\phi] \\ &= \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_2 l\dot{x}\dot{\phi}\cos\phi + \frac{1}{2}m_2 l^2\dot{\phi}^2 - m_2 gl\cos\phi. \end{aligned}$$

Because we do not have explicit time dependence in the Lagrangian, our Hamiltonian is constant:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}t} = 0.$$

The last thing to notice is that the constant Hamiltonian is equivalent with the conservation law for energy. Because our potential (and constrains) is independent of velocity and time the Hamiltonian is the same as the total energy in the system. One can confirm this from previous exercise 4.1 where we have

$$T = T_1 + T_2 = \frac{1}{2}m_1\dot{x} + \frac{1}{2}m_2[\dot{x}^2 + 2l\dot{x}\dot{\phi}\cos\phi + l^2\dot{\phi}^2]$$

and

$$V = -m_2 gl\cos\phi.$$

3. **Solution**:
The Lagrangian is

$$L = -mc^2\sqrt{1-\frac{\dot r^2 + r^2\dot\phi^2}{c^2}} + \frac{k}{r}.$$

It is good to notice that this Lagrangian is not similar to the others in this course $L \neq T - V$. This is because in the relativistic case one has to go a little bit different way to variate the action when one uses the Hamilton principle. More about this in the theoretical course, Classical Field Theory. Now the momenta are

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = \frac{mc^2}{\sqrt{1-\frac{\dot r^2+r^2\dot\phi^2}{c^2}}}\cdot\frac{r^2\dot\phi}{c^2} = \frac{mr^2\dot\phi}{\sqrt{1-\frac{\dot r^2+r^2\dot\phi^2}{c^2}}}$$

$$p_r = \frac{\partial L}{\partial\dot r} = \frac{mc^2}{\sqrt{1-\frac{\dot r^2+r^2\dot\phi^2}{c^2}}}\cdot\frac{\dot r}{c^2} = \frac{m\dot r}{\sqrt{1-\frac{\dot r^2+r^2\dot\phi^2}{c^2}}}.$$

Because of $\partial_\phi L = 0$, momentum p_ϕ is a constant of motion but p_r is not:

$$\frac{\partial L}{\partial r} = \frac{mc^2}{\sqrt{1-\frac{\dot r^2+r^2\dot\phi^2}{c^2}}}\cdot\frac{r\dot\phi^2}{c^2} - \frac{k}{r^2}.$$

Let's denote $E_0 = mc^2$ and

$$\gamma = \left[\sqrt{1-\frac{\dot r^2 + r^2\dot\phi^2}{c^2}}\right]^{-1}.$$

Thus

$$p_r = \gamma E_0\cdot\frac{\dot r}{c^2} \quad \text{and} \quad p_\phi = \gamma E_0\cdot\frac{r^2\dot\phi}{c^2}$$

and also

$$\frac{\partial L}{\partial r} = \gamma E_0\cdot\frac{r\dot\phi^2}{c^2} + \frac{V(r)}{r}.$$

Also the Lagrangian looks more simple

$$L = -\gamma^{-1}E_0 + \frac{k}{r}.$$

The Lagrange equations are

$$\frac{\mathrm{d}p_\phi}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}p_r}{\mathrm{d}t} = \gamma E_0 \cdot \frac{r\dot{\phi}^2}{c^2} + \frac{V(r)}{r}.$$

Because $\partial_t L = 0 \Rightarrow \dot{H} = 0$, our Hamiltonian is conserved. By definition

$$\begin{aligned}
H &= \sum_i \dot{q}_i p_i - L \\
&= \gamma E_0 \frac{\dot{r}^2}{c^2} + \gamma E_0 \frac{r^2\dot{\phi}^2}{c^2} - [-\gamma^{-1} E_0 + \frac{k}{r}] \\
&= \gamma E_0 \Big(\frac{\dot{r}^2 + r^2\dot{\phi}^2}{c^2} \Big) + \gamma^{-1} E_0 - \frac{k}{r} \\
&= \gamma E_0 (1 - \gamma^{-2}) + \gamma^{-1} E_0 - \frac{k}{r} \\
&= \gamma E_0 - \frac{k}{r}
\end{aligned}$$

that agrees with relativity, like the momenta do. In the classical limit $v << c$ the Hamilton reduces to the classical case

$$H = \gamma E_0 - \frac{k}{r} \approx mc^2 + \frac{1}{2}mv^2 - \frac{k}{r}.$$

Because the potential is independent on time and velocity, the classical Hamiltonian should be the total energy. This agrees with the relativistic Hamiltonian when one gets to classical limit: kinetic energy is the normal

$$T = \frac{1}{2}mv^2$$

but the potential energy also include the rest energy

$$V_{\text{class}} = mc^2 - \frac{k}{r} = E_0 + V.$$

4. **Solution**:
Kinetic energy in spherical coordinates is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2 \sin^2\theta + r^2\dot{\theta}^2).$$

In the system it holds

$$r = a = \text{constant} \Rightarrow \dot{r} \equiv 0$$
$$\dot{\phi} = \omega = \text{constant} \Rightarrow \phi = \omega t + \phi_0$$

meaning that the system has kinetic energy

$$T = \frac{1}{2}m(a^2\omega^2 \sin^2\theta + a^2\dot{\theta}^2) = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta$$

Potential energy is

$$V = mga\cos\theta.$$

Now the Lagrangian is

$$L = T - V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta$$

The total energy is

$$E = T + V = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta.$$

For the Hamiltonian one needs to define canonical momentum

$$p = \frac{\partial L}{\partial\dot{\theta}} = ma^2\dot{\theta}.$$

From the definition the Hamiltonian is

$$\begin{aligned} H &= \sum_i \dot{q}_i p_i - L \\ &= \dot{\theta}p - L \\ &= ma^2\dot{\theta}^2 - (\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2\omega^2\sin^2\theta - mga\cos\theta) \\ &= \frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta. \end{aligned}$$

Now one can clearly see that the Hamiltonian is not the same as the total energy in the system $H \neq E$. Because the Lagrangian does not explicitly depend on time, *the Hamiltonian is conserved*:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow \frac{\mathrm{d}H}{\mathrm{d}t} = 0.$$

Due to the constraint in the system, $r = a =$ constant and $\dot{\phi} = \omega =$ constant one ends up with the Lagrangian that does not depend on time (explicitly). As mentioned in the lectures, in this kind of case one has a conservation law $H =$ constant, but the Hamilton does not have to be same as the total energy. Let's prove this conclusion:

$$\begin{aligned}\frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}(\frac{1}{2}ma^2\dot{\theta}^2 - \frac{1}{2}ma^2\omega^2\sin^2\theta + mga\cos\theta) \\ &= ma^2\ddot{\theta}\dot{\theta} - a^2\omega^2\sin\theta\cos\theta\dot{\theta} - mga\sin\theta\dot{\theta} \\ &= \dot{\theta}\underbrace{[ma^2\ddot{\theta} - a^2\omega^2\sin\theta\cos\theta - mga\sin\theta]}_{=0} \\ &= 0,\end{aligned}$$

where we use the Lagrange equation

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \\ &\Leftrightarrow \\ &ma^2\ddot{\theta} - a^2\omega^2\sin\theta\cos\theta - mga\sin\theta = 0.\end{aligned}$$

It is easy to see that

$$\frac{\mathrm{d}E}{\mathrm{d}t} \neq 0.$$

This implies that the energy in the system is not conserved. We still have the conservation law of energy? The solution is that our system is not isolated but it exchanges energy with the environment. The environment is acting to the system with the force that keeps the angular velocity as constant (note that our Hamiltonian regards this). Notice that the combined system (environment + our system) is isolated meaning

$$E_{\text{system}} + E_{\text{enviroment}} = \text{constant} + H.$$

So our conservation law of energy holds in the combined system.