1. **Solution**:
In the lectures we have the formula

$$\tau = \frac{2\pi}{\sqrt{G(m_1 + m_2)}} a^{3/2}$$

but if $m_1 >> m_2$, we can approximate

$$\tau \approx \frac{2\pi}{\sqrt{Gm_1}} a^{3/2}.$$

Now we use this formula for two different cases:

$$\tau_{\text{EM}} \approx \frac{2\pi}{\sqrt{Gm_E}} a_{\text{EM}}^{3/2} \quad \text{the Earth-Moon pair}$$

$$\tau_{\text{ES}} \approx \frac{2\pi}{\sqrt{Gm_S}} a_{\text{ES}}^{3/2} \quad \text{the Earth-Sun pair.}$$

Thus the masses are

$$m_E = \frac{4\pi^2}{G\tau_{\text{EM}}^2} a_{\text{EM}}^3$$

$$m_S = \frac{4\pi^2}{G\tau_{\text{ES}}^2} a_{\text{ES}}^3$$

and now it is easy to calculate the ratio of the masses

$$\frac{m_E}{m_S} = \left(\frac{\tau_{\text{ES}}}{\tau_{\text{EM}}}\right)^2 \left(\frac{a_{\text{EM}}}{a_{\text{ES}}}\right)^3.$$

Using given values results to

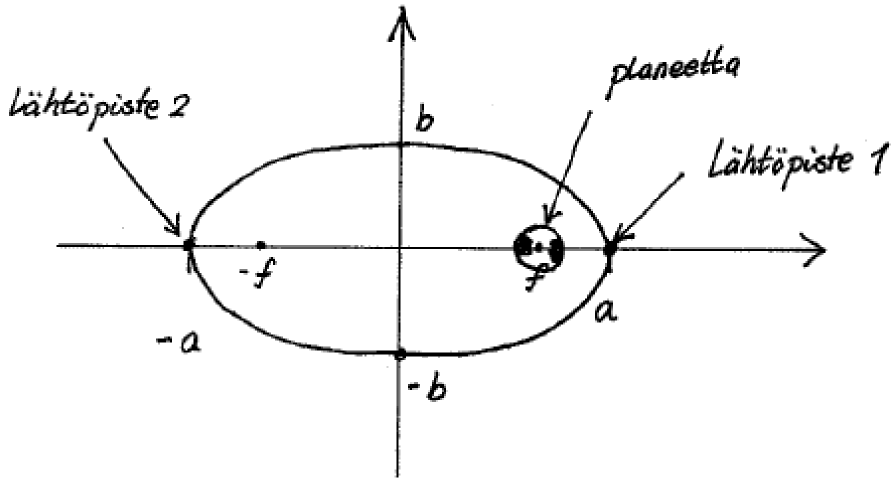
$$\frac{m_E}{m_S} \approx 2.97 * 10^{-6}$$

or in a similar way

$$\frac{m_S}{m_E} \approx 337000 \quad (\text{real value} \approx 333000).$$

2. **Solution**:
The satellite either starts from the point 1 or 2 ($\pm a$) and it has gravitional potential

$$V = -\frac{k}{m} = -\frac{GMm}{r}.$$



We know the eccentricity of the orbit

$$\epsilon = \sqrt{1 + \frac{2El^2}{m'k^2}}$$

where m' is the reduced mass

$$m' = \frac{mM}{m+M} = \frac{m}{\frac{m}{M}+1} \approx m, \quad m << M$$

and l is the angular momentum that is conserved, meaning that

$$l = l_{\text{beginning}} = mrv.$$

Now the eccentricity has a form

$$\begin{aligned} \epsilon &= \left[1 + \frac{2(\frac{1}{2}mv^2 - \frac{GMm}{r})(mrv)^2}{m(GMm)^2}\right]^{1/2} \\ &= \left[1 - 2\frac{rv^2}{GM} + \frac{r^2v^4}{G^2M^2}\right]^{1/2} \\ &= \sqrt{\left(1 - \frac{rv^2}{GM}\right)^2} \\ &= \left|1 - \frac{rv^2}{GM}\right|. \end{aligned}$$

We have different orbits depending on different eccentricities:
circle

$$\epsilon = 0 \Rightarrow 1 - \frac{rv^2}{GM} = 0 \Rightarrow v = \sqrt{\frac{GM}{r}},$$

parabola

$$\epsilon = 1 \Rightarrow 1 - \frac{rv^2}{GM} = \pm 1 \Rightarrow v = \sqrt{\frac{2GM}{r}} \quad \text{or} \quad v = 0,$$

and *hyperbola*

$$\begin{aligned}
&\epsilon > 1 \\
&\Rightarrow \underbrace{1 - \underbrace{\frac{rv^2}{GM}}_{>0}}_{<1} > 1 \quad \text{or} \quad 1 - \frac{rv^2}{GM} < -1 \\
&\Rightarrow \frac{rv^2}{GM} > 2 \\
&\Rightarrow v > \sqrt{\frac{2MG}{r}}.
\end{aligned}$$

For *an elliptic orbit* the velocity has to be in the interval

$$0 < v < \sqrt{\frac{2MG}{r}}.$$

Because the case $v = \sqrt{\frac{GM}{R}}$ is the circle orbit, the starting point of the satellite is point 1, if $\sqrt{MG/r} < v < \sqrt{2MG/r}$, and point 2, if $0 < v < \sqrt{MG/r}$.

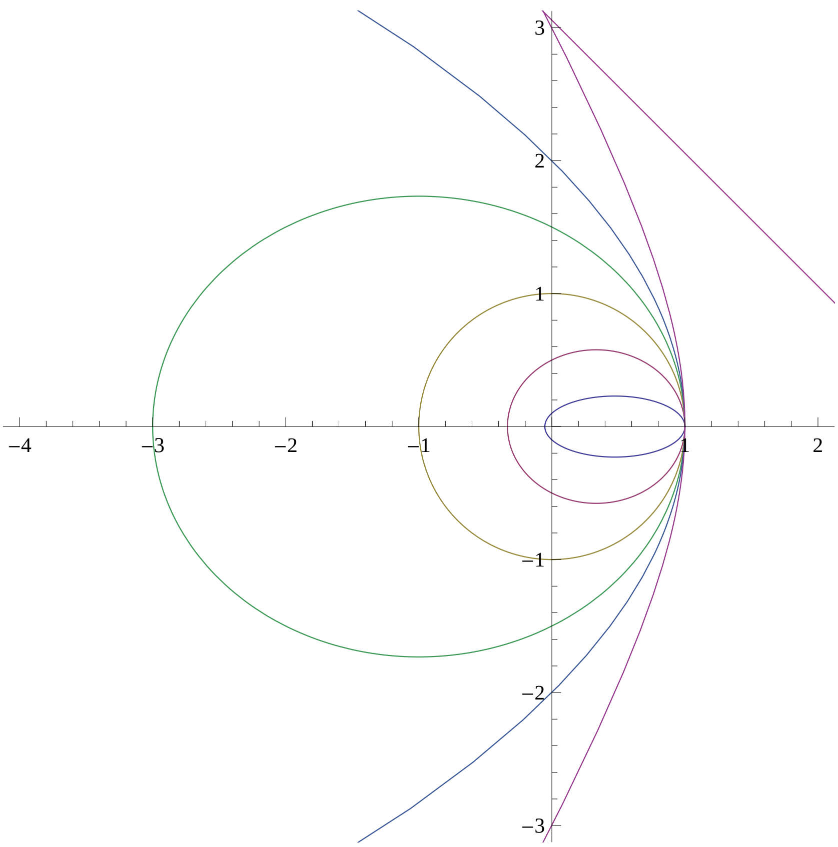


Figure 1: *Different orbits*

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3. **Solution**:

We start from the formula in the lectures

$$\phi = \phi_0 \pm \frac{l}{\sqrt{2m}} \int_{r_0}^{r} \frac{\mathrm{d}r}{r^2\sqrt{E - V_{\text{eff}}}}. \tag{1}$$

Now we are interested in an angular difference $\Delta\phi$ that is the angle of the two sequential extreme values of r. Let's substitute $r = r_{\text{max}}$ and $r_0 = r_{\text{min}}$:

$$\Delta\phi = \phi - \phi_0 = \frac{l}{\sqrt{2m}} \int_{r_{\text{min}}}^{r_{\text{max}}} \frac{\mathrm{d}r}{r^2\sqrt{E - V_{\text{eff}}}}. \tag{2}$$

The next problem is to find the extrema values of r. Naturally these are points, where $E - V_{\text{eff}}$ vanish (see the lectures). Let's assume that r is close to r_0. In this environment we can expand the potential V_{eff} as Taylor series up to second-order

$$V_{\text{eff}}(r) \approx V_{\text{eff}}(r_0) + \underbrace{V_{\text{eff}}'(r_0)}_{=0}(r - r_0) + \frac{1}{2}V_{\text{eff}}''(r_0)(r - r_0)^2$$

and thus

$$E - V_{\text{eff}}(r) \approx E - \underbrace{V_{\text{eff}}(r_0))}_{\equiv E_0} - \frac{1}{2}V_{\text{eff}}''(r_0)(r - r_0)^2.$$

Now it is easy to solve

$$E - V_{\text{eff}}(r_{\text{min,max}}) = 0$$

meaning

$$r_{\text{min,max}} = r_0 \pm \sqrt{\frac{2(E - E_0)}{V_{\text{eff}}''(r_0)}}.$$

So our integration limits are for now checked. After this let's look the integrand. Because

$$r\sqrt{E - E_0} = r_0\sqrt{E - E_0} + \underbrace{(r - r_0)}_{\text{small}}\underbrace{\sqrt{E - E_0})}_{\text{small}} \approx r_0\sqrt{E - E_0}$$

we can replace r with r_0. In the above we approximate that small times small is zero. Our integral gets easier with changing variables:

$$r - r_0 = s\sqrt{\frac{2(E - E_0)}{V_{\text{eff}}''(r_0)}}.$$

Now we have to check again the limits of the integral: when $r = r_{\text{max}}$, it is clearly $s = 1$ and when $r = r_{\text{min}}$ implies $s = -1$. Furthermore we have

$$\mathrm{d}r = \sqrt{\frac{2(E - E_0)}{V_{\text{eff}}''(r_0)}}\mathrm{d}s$$

and

$$E - V_{\text{eff}}(r) = (E - E_0)(1 - s^2).$$

After all this we get

$$\begin{aligned}
\Delta\phi &= \frac{l}{\sqrt{2m}} \int_{-1}^{1} \frac{1}{r_0^2} \frac{1}{\sqrt{E - E_0}} \frac{1}{\sqrt{1 - s^2}} \sqrt{\frac{2(E - E_0)}{V''_{\text{eff}}(r_0)}} \mathrm{d}s \\
&= \frac{l}{r_0^2 \sqrt{m V''_{\text{eff}}(r_0)}} \int_{-1}^{1} \frac{\mathrm{d}s}{\sqrt{1 - s^2}} \\
&= \frac{l\pi}{r_0^2 \sqrt{m V''_{\text{eff}}(r_0)}}
\end{aligned}$$

where the last integral can be solved with e.g. change of variables $s = \sin\alpha$. Now we fix the potential and start to study a potential that has a form $V(r) = ar^{n-1}$. This means that

$$V_{\text{eff}}(r) = \frac{l^2}{2mr^2} + ar^{n+1}.$$

We can also calculate minimum at the point r_0 where $V'_{\text{eff}}(r_0) = 0$ meaning

$$V'_{\text{eff}}(r) = -2\frac{l^2}{2mr^3} + a(n+1)r^n.$$

The most useful thing is to solve the constant a

$$a = \frac{l^2}{(n+1)mr_0^{n+3}}.$$

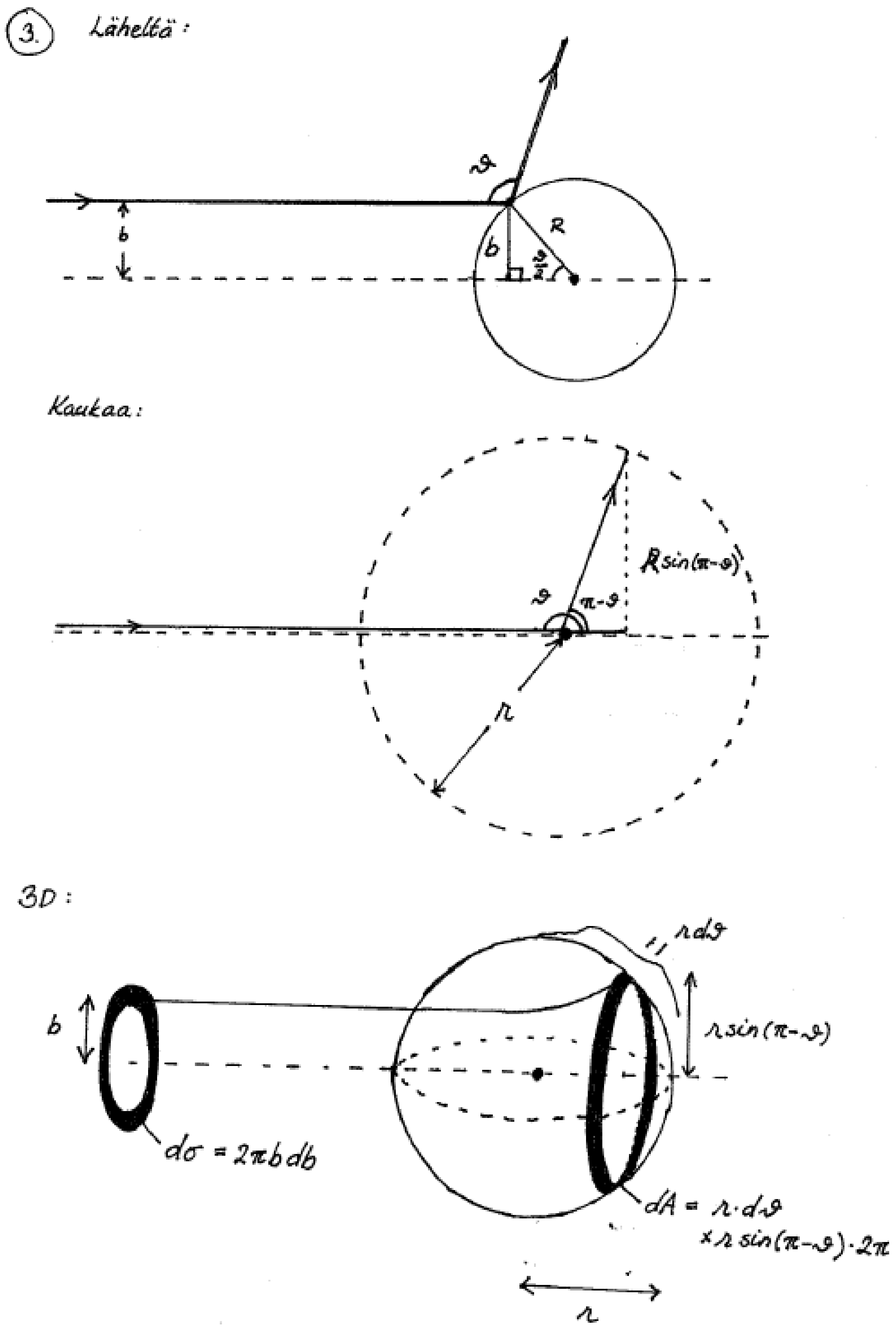
The only thing we still need is $V''_{\text{eff}}(r_0)$:

$$\begin{aligned}
V''_{\text{eff}}(r_0) &= \frac{3l^2}{mr_0^4} + a(n+1)nr_0^{n-1} \\
&= \frac{3l^2}{mr_0^4} + \frac{l^2}{(n+1)mr_0^{n+3}}(n+1)nr_0^{n-1} \\
&= \frac{3l^2}{mr_0^4} + \frac{l^2 n}{mr_0^4} \\
&= \frac{l^2}{mr_0^4}(n+3).
\end{aligned}$$

By substituting our result into the equation of $\Delta\phi$ we get

$$\Delta\phi = \frac{\pi}{\sqrt{3+n}}.$$

4. **Solution**:



3.
Läheltä:
b
R
Kaukaa:
R sin(π-ϑ)
ϑ
π-ϑ
r
3D:
b
dσ = 2πb db
r dϑ
r sin(π-ϑ)
dA = r·dϑ
× r sin(π-ϑ)·2π
r

From the picture we notice that using trigonometic identies we get

$$\sin\frac{\theta}{2} = \frac{b}{R}$$

and differentiating this leads us to (*)

$$\begin{aligned}\frac{1}{2}\cos\frac{\theta}{2}\mathrm{d}\theta &= \frac{\mathrm{d}b}{R}\\ \Rightarrow \mathrm{d}\theta &= \frac{2\mathrm{d}b}{R\cos\frac{\theta}{2}}.\end{aligned}$$

From the interval $[b, b+\mathrm{d}b]$, meaning that particles coming from the area $2\pi b\mathrm{d}b$ scatter to an angle $[\theta, \theta+\mathrm{d}\theta]$ and thus they are leaving from the scattering center through a circle that has an area

$$\mathrm{d}A = r\mathrm{d}\theta * 2\pi r\sin(\pi-\theta).$$

So the solid angle where the particles are scattering is

$$\begin{aligned}\mathrm{d}\Omega &= \frac{\mathrm{d}A}{r^2}, \quad \text{the definition of the solid angle}\\ &= 2\pi\sin(\pi-\theta)\mathrm{d}\theta, \quad \sin(\pi-\alpha)=\alpha \quad \forall\alpha\\ &= 2\pi\sin\theta\mathrm{d}\theta, \quad \text{we know } \mathrm{d}\theta\\ &= 2\pi\sin\theta\frac{2\mathrm{d}b}{R\cos\frac{\theta}{2}}, \quad \sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\\ &= 8\pi\sin\frac{\theta}{2}\frac{\mathrm{d}b}{R}.\end{aligned}$$

Furthermore, $\mathrm{d}\sigma = 2\pi b\mathrm{d}b = 2\pi R\sin(\theta/2)\mathrm{d}b$. Now the differential cross section is

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{2\pi R\sin(\theta/2)\mathrm{d}b}{8\pi\sin\frac{\theta}{2}\frac{\mathrm{d}b}{R}} = \frac{R^2}{4}.$$

and the total cross section is

$$\sigma = \int_\Omega \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\frac{R^2}{4}\sin\theta\mathrm{d}\theta\mathrm{d}\phi = \frac{R^2}{4} * 4\pi = \pi R^2.$$

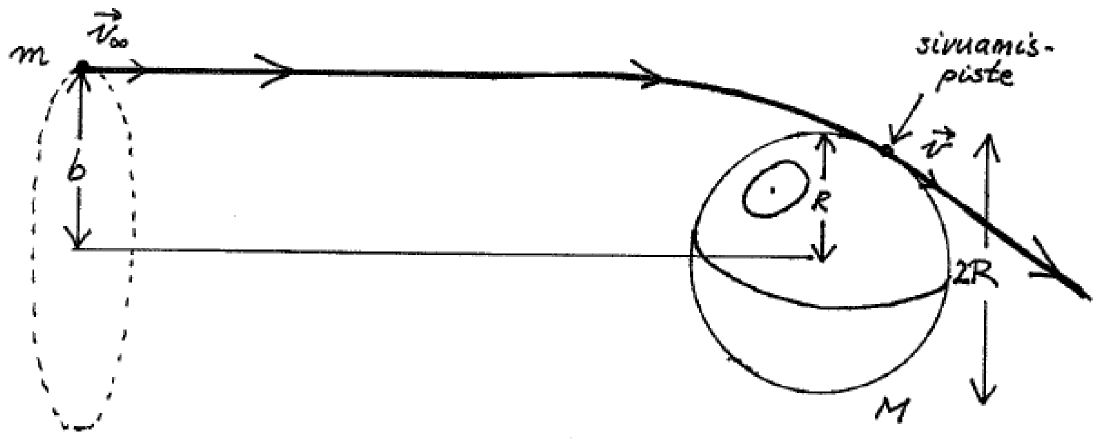
The second way of calculating the differential cross section is to notice from the begin (*)

$$\frac{\mathrm{d}b}{\mathrm{d}\theta} = \frac{1}{2}R\cos\frac{\theta}{2}$$

and use the formula in the lectures

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| = \frac{1}{2}R^2\frac{\cos\frac{\theta}{2}\sin\frac{\theta}{2}}{\sin\theta} = \frac{R^2}{4}.$$

5. **Solution**:



The collision cross section is the area that a particle has to go through to collide with the scatterer. Here our scatterer is a sphere. Thus the area through which a particle collides is a disk meaning that it can be presented as

$$\sigma_{\text{col}} = \pi b^2.$$

So our only problem is to solve what is b. Without gravity it is clear that b would be R but now we have to notice gravity that pulls our particle towards our sphere. Our requirement is that a particle leaving with the impact parameter b will touch the sphere. Far, far a way from the sphere gravitational potential is zero (approximately) and the particle has energy

$$E = \frac{1}{2} m v_\infty^2.$$

The angular momentum is

$$L = m v_\infty b.$$

On the other hand in the point of the contact the gravitational potential is

$$V = -\frac{GmM}{R}$$

and the energy is thus

$$E = \frac{1}{2} m v^2 - \frac{GmM}{R}.$$

In the point of contact the angular momentum is

$$L = mvR.$$

Because the angular momentum is a conserved, we have a relation

$$mv_\infty b = mvR \Rightarrow v = \frac{b}{R}v_\infty.$$

Also particle's energy is conserved quantity:

$$\begin{aligned}
&\frac{1}{2}mv_\infty^2 = \frac{1}{2}mv^2 - \frac{GmM}{R} \\
&\Leftrightarrow \\
&b^2 = R^2 + \frac{2GMR}{v_\infty^2}.
\end{aligned}$$

So the cross section is

$$\sigma_{\text{col}} = \pi b^2 = \pi R^2 + \frac{2\pi GMR}{v_\infty^2}.$$

This can be done also with the lecture notes. The hint tells that $f - a = R \Rightarrow f = R + a$. From the lectures one gets

$$\begin{aligned}
b^2 &= f^2 - a^2 \\
&= (R+a)^2 - a^2 \\
&= R^2 - 2Ra \\
&= R^2 + \frac{R|k|}{E} \\
&= R^2 + \frac{2R|k|}{mv_\infty^2} \\
&= R^2 + \frac{2RGM}{v_\infty^2},
\end{aligned}$$

where one uses from the lectures

$$a = \frac{|k|}{2E}, \quad E = \frac{1}{2}mv_\infty^2 \quad \text{and} \quad |k| = GmM.$$