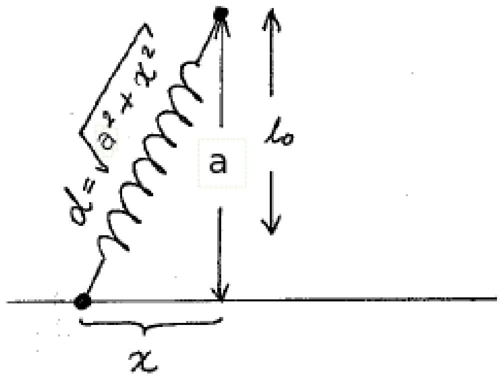
1. **Solution**:



The kinetic energy of the system is

$$T = \frac{1}{2}m\dot{x}^2$$

and potential energy is

$$V = \frac{1}{2}k(d - l_0)^2 = \frac{1}{2}k(\sqrt{a^2 + x^2} - l_0)^2.$$

This means that the Lagrangian for the system is

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(\sqrt{a^2 + x^2} - l_0)^2.$$

In the limit of small oscillations meaning $|x| << 1$ we can approximate

$$\sqrt{a^2 + x^2} \approx a + \frac{1}{2}\frac{x^2}{a}$$

and thus the Lagrangian simplifies as

$$\begin{aligned} L &\approx \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k(a + \frac{1}{2}\frac{x^2}{a} - l_0)^2 \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\left[(a - l_0)^2 + 2(a - l_0)\frac{x^2}{2a} + \underbrace{\frac{1}{4}\frac{x^4}{a^2}}_{\approx 0}\right] \\ &= \frac{1}{2}m\dot{x}^2 - \frac{1}{2}k\left(1 - \frac{l_0}{a}\right)x^2 - V_0 \end{aligned}$$

where we denote $V_0 = \frac{1}{2}k(a - l_0)^2$. The force in the point $x = 0$ is $F = k(a - l_0)$ implying

$$k(1 - \frac{l_0}{a}) = \frac{F}{a}.$$

Now we can write the Lagrangian of small oscillations as

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}\frac{F}{a}x^2$$

where we neglect V_0 as a constant (does not affect on the equation of motion). The Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$
$$\Leftrightarrow$$
$$m\ddot{x} + \frac{F}{a}x = 0$$

that is again the equation of the harmonic oscillator. The solution is

$$x = A\cos(\omega t + \delta), \quad A \text{ and } \delta \text{ are constant}$$

where

$$\omega^2 = \frac{F}{ma}.$$

This can be also calculated using the lecture notes. From the notes we know that the Lagrangian for small oscillations is

$$L = \frac{1}{2}\sum_{ij}[A_{ij}\dot{\eta}_i\dot{\eta}_j - v_{ij}\eta_i\eta_j] - V_0.$$

In our case $i = j = 1$ and the matrices reduce to scalar numbers. Comparing the Lagrangians we get $A = m$ and $v = F/a$. Now using the equation in the lectures we get the same answer as before:

$$v - \omega^2 A = 0 \Leftrightarrow \omega^2 = \frac{v}{A} = \frac{F}{ma}.$$

2. **Solution**:
Let's denote the distance of the attachment points with d and the distance of the pendulum masses with s. The location of pendulum 1 is

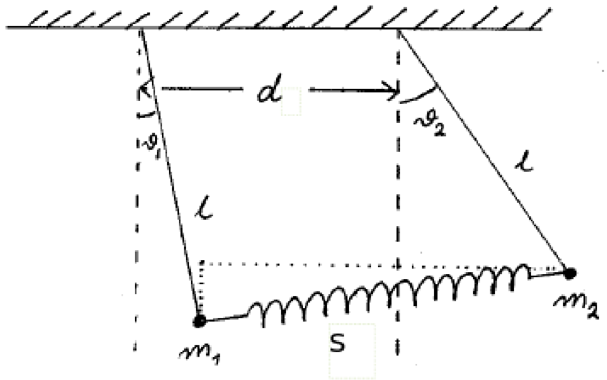
$$\mathbf{r}_1 = l\sin\theta_1\mathbf{i} - l\cos\theta_1\mathbf{j}$$

and pendulum 2 has

$$\mathbf{r}_2 = (d + l\sin\theta_2)\mathbf{i} - l\cos\theta_2\mathbf{j}.$$

So the system has kinetic energy

$$T = \frac{1}{2}m\dot{\mathbf{r}}_1^2 + \frac{1}{2}m\dot{\mathbf{r}}_2^2 = \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$



and potential energy

$$V = \frac{1}{2}k(s-d)^2 - mgl\cos\theta_1 - mgl\cos\theta_2,$$

where

$$s = \sqrt{(d + l\sin\theta_2 - l\sin\theta_1)^2 + (l\cos\theta_1 - l\cos\theta_2)^2}.$$

Thus the Lagrangian is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}ml^2(\dot\theta_1^2 + \dot\theta_2^2) + mgl(\cos\theta_1 + \cos\theta_2) \\ &- \frac{1}{2}k(\sqrt{(d + l\sin\theta_2 - l\sin\theta_1)^2 + (l\cos\theta_1 - l\cos\theta_2)^2} - d)^2. \end{aligned}$$

In the limit of small oscillations we can make a series representation of the Lagrangian with respect to θ_1 and θ_2. We only take second-order terms into account. Let the new parameters be $\eta_1 = l\sin\theta_1 \approx l\theta_1$ and $\eta_2 = l\sin\theta_2 \approx l\theta_2$. We also need to know that $\cos\theta \approx 1 - \frac{1}{2}\theta^2$. Now the distance d is

$$\begin{aligned} s^2 &= (d + l\sin\theta_2 - l\sin\theta_1)^2 + (l\cos\theta_1 - l\cos\theta_2)^2 \\ &\approx (d + l\theta_2 - l\theta_1)^2 + l^2(1 - \frac{1}{2}\theta_1^2 - 1 + \frac{1}{2}\theta_2^2)^2 \\ &= d^2 + 2dl(\theta_2 - \theta_1) + l^2(\theta_2-\theta_1)^2 + \underbrace{l^2\left(\frac{1}{2}\theta_2^2 - \frac{1}{2}\theta_1^2\right)^2}_{\approx 0} \\ &\approx d^2 + 2dl(\theta_2 - \theta_1) + l^2(\theta_2-\theta_1)^2 \\ &= [d + l(\theta_2 - \theta_1)]^2 \end{aligned}$$

meaning that

$$s \approx d + l(\theta_2 - \theta_1) = d + \eta_2 - \eta_1.$$

Now we can see that in this case the approximation for small oscillations means that we ignore the motion of the system in y-direction. The gravitational potential has an approximate form as

$$-mgl(\cos\theta_1 + \cos\theta_2) \approx -mgl\left(1 - \frac{1}{2}\theta_1^2 + 1 - \frac{1}{2}\theta_2^2\right) = \frac{mg}{2l}(\eta_1^2 + \eta_2^2) - \underbrace{2mgl}_{=V_0}$$

where we can neglect V_0 because it is just a constant (scaling factor). After this the Lagrangian for small oscillations is

$$\begin{aligned}
L &= \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{mg}{2l}(\eta_1^2 + \eta_2^2) - \frac{1}{2}k(d + \eta_2 - \eta_1 - d)^2 \\
&= \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{mg}{2l}(\eta_1^2 + \eta_2^2) - \frac{1}{2}k(\eta_2^2 + \eta_1^2 - 2\eta_1\eta_2) \\
&= \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \left(\frac{mg}{2l} + \frac{k}{2}\right)(\eta_1^2 + \eta_2^2) + k\eta_1\eta_2 \\
&= \frac{1}{2}m(\dot{\eta}_1^2 + \dot{\eta}_2^2) - \frac{1}{2}\left(\frac{mg}{l} + k\right)(\eta_1^2 + \eta_2^2) + k\eta_1\eta_2.
\end{aligned}$$

3. **Solution**:
First we present the Lagrangian in the form of

$$L = \frac{1}{2}\sum_{ij} A_{ij}\dot{x}_i\dot{x}_j - \frac{1}{2}\sum_{ij} v_{ij}x_ix_j.$$

For two variables η_1 and η_2 this is

$$L = \frac{1}{2}A_{11}\dot{\eta}_1^2 + \frac{1}{2}A_{12}\dot{\eta}_1\dot{\eta}_2 + \frac{1}{2}A_{22}\dot{\eta}_2^2 - \frac{1}{2}v_{11}\eta_1^2 - \frac{1}{2}v_{12}\eta_1\eta_2 - \frac{1}{2}v_{22}\eta_2^2$$

where we used the fact that the matrices A_{ij} and v_{ij} are symmetric meaning $v_{12} = v_{21}$ and $A_{12} = A_{21}$. The symmetry allows us to combine the cross terms. Comparing this Lagrangian to the Lagrangian in the previous problem we get

$$A = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix}.$$

To solve the eigenfrequencies we have to solve the eigenvalue problem

$$\det\big(v - \omega^2 A\big) = 0$$

$$\Leftrightarrow$$

$$\left| \begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix} - \omega^2 \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \right| = 0$$

$$\Leftrightarrow$$

$$\left| \begin{pmatrix} \frac{mg}{l} - m\omega^2 + k & -k \\ -k & \frac{mg}{l} + k - m\omega^2 \end{pmatrix} \right| = 0$$

$$\Leftrightarrow$$

$$\left(\frac{mg}{l} + k - m\omega^2\right)^2 - k^2 = 0$$

$$\Leftrightarrow$$

$$\frac{mg}{l} + k - m\omega^2 = \pm k$$

$$\Leftrightarrow$$

$$\omega^2 = -\frac{1}{m}\left(\pm k - k - \frac{mg}{l}\right)$$

$$\Leftrightarrow$$

$$\omega^2 = \frac{g}{l} \quad \text{or} \quad \omega^2 = \frac{2k}{m} + \frac{g}{l}.$$

Because we have two eigenvalues we will have two different, linearly independent eigenvectors that are achieved by solving the equation

$$(v - \omega^2 A)X = 0.$$

If we choose first (case 1)

$$\omega^2 = \frac{g}{l} \quad \text{and} \quad X = X_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$$

we have

$$(v - \omega^2 A)X_1 = 0$$

$$\Leftrightarrow$$

$$\left[\begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix} - \begin{pmatrix} \frac{mg}{l} & 0 \\ 0 & \frac{mg}{l} \end{pmatrix} \right] \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0$$

$$\Leftrightarrow$$

$$k \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0$$

$$\Leftrightarrow$$

$$x_{11} = x_{12}.$$

Let's choose $x_{11} = x_{12} = \frac{1}{\sqrt{2}}$ so that $x_{11}^2 + x_{12}^2 = 1$ and then

$$X_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

If we choose on the other hand (case 2)

$$\omega^2 = \frac{2k}{m} + \frac{g}{l} \quad \text{and} \quad X = X_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}.$$

we have

$$\begin{aligned}
&(v - \omega^2 A)X_2 = 0 \\
&\Leftrightarrow \\
&\left[\begin{pmatrix} \frac{mg}{l} + k & -k \\ -k & \frac{mg}{l} + k \end{pmatrix} - \begin{pmatrix} 2k + \frac{mg}{l} & 0 \\ 0 & 2k + \frac{mg}{l} \end{pmatrix} \right] \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = 0 \\
&\Leftrightarrow \\
&k \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = 0 \\
&\Leftrightarrow \\
&x_{21} = -x_{22}.
\end{aligned}$$

Let's choose $x_{11} = -x_{12} = \frac{1}{\sqrt{2}}$ and then

$$X_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The physical interpretation is that in the case 1 the pendulums are oscillating to the same direction and with same amplitude. Thus the spring does not affect on the oscillation of the system. In the case 2 the pendulums are oscillating towards each other and the spring matters.