1. **Fast Fourier transformation.** Calculate the discrete Fourier transform of the function

$$f(x) = 1 + 2\sin x + 3\cos 2x$$

using the Danielson–Lanczos algorithm. Do this first by hand. You need to tabulate values of the function f on a number of points that is a power of two, say $N = 2^p$. Start from the definition of the discrete transform ans subdivide the summation as described in the lecture notes into even and odd parts, F_k^e and F_k^o,

$$\begin{aligned} F_k &= \sum_{j=0}^{N-1} \exp\left(2\pi ijk/N\right) f_j \\ &= \sum_{j=0}^{N/2-1} \exp\left(2\pi i(2j)k/N\right) f_{2j} + \sum_{j=0}^{N/2-1} \exp\left(2\pi i(2j-1)k/N\right) f_{2j-1} \\ &= F_k^e + W^k F_k^o, \qquad W = e^{2\pi i/N}, \end{aligned}$$

where f_i:s are the tabulated values and $k = 0, \ldots, N-1$. For example,

$$F_k^e = F_k^{ee} + W^{2k} F_k^{eo} \tag{1}$$

After p divisions the summations can no longer be subdivided. Relate the terms $F_k^{eeoo\cdots}$ and the tabulated values using the bit-reversal method described in the lectures. After doing the neccessary algebra, you can evaluate F_k, $k = 0, \ldots, N-1$.

Compare your result to the one given by Numerical Recipes routine `four1`. Study the effect of changing the sampling rate and interval.

2. **FFT of an Gaussian function.** Calculate the FFT, power spectral density and total power of function

$$g(t) = \exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)$$

Study the behaviour of the function in time- and frequency domain as parameters t_0 and σ are varied.

3. **Correlation using FFT.** Calculate the correlation between the Gaussian function g, as defined in previous problem, and $h(t) = g(t + t_{\text{lag}}) + \alpha\eta(t)$, where η produces random noise with an amplitude of 1. Coefficient α scales the noise level and t_{lag} effectively shifts the peak of g.

Correlation $\text{Corr}(g, h)$ can be calculated using the Fourier transform $\mathcal{F}$:

$$\mathcal{F}(g) \equiv G, \qquad \mathcal{F}(h) \equiv H$$

$$\mathcal{F}(\text{Corr}(g,h)) = G(f)H(-f) \overset{g,h \text{ real}}{=} G(f)H^*(f)$$