Cosmology: Homework 1. Solutions.

1: Olber's paradox.

We have an inifinite, eternal, unchanging Universe. Consider a spherical shell of thickness dr and radius r. There are

$$4\pi r^2 dr n_*$$

stars in it, covering a fraction

$$dr n_* r_\odot^2 \pi$$

of the sky. Integrating all these spherical shells up to a distance R_{max}, we get a "fraction of angle"

$$\pi r_\odot^2 R_{max} n_*.$$

This

- a) eventually becomes larger than 1, ie stars cover the whole sky,
- b) goes to a half when $R_{max} = \frac{1}{2\pi r_\odot^2 n_*} = 3.1 \times 10^{18}$Mpc.,
- c) means that if the Universe is instead 4.6×10^9 years old, only a fraction $\pi r_\odot^2 R_{max} n_* = 2.3 \times 10^{-16}$ of the sky is covered by stars. The energy density of radiation would be age×power×density of stars which translates into 2.1×10^{-33}kg/m^3.
- d) For galaxies, the same equation gives 50 percent coverage for $R_{max} = 5.3 \times 10^5$Mpc.

The equation above does not actually make sense for large values of R_{max}, sice the coverage can never go above 1. This is because we haven't taken into account that stars will start shading off each other.

Note the more elegant solution (thank you, Timo!) which avoids this problem: consider the cylinder of length r with crosssection $\pi r_\odot^2$. If there are no stars inside that cylinder, the line of sight corresponding to the axis of the cylinder does not meet the surface of a star.

Inside the volume $4\pi/3r^3$, there are $n_* 4\pi/3r^3$ stars, randomly distributed, and so the probability of *not* having any inside the cylinder is

$$\left(1 - \frac{\pi r_\odot^2 r}{4\pi/3r^3}\right)^{n_* 4\pi/3r^3}.$$

For large value of r, this approaches the exponential function

$$\to e^{\pi r_\odot^2 n_* r},$$

and so the probability of having a star in the cylinder is

$$P(\text{star in cylinder}) = 1 - e^{-\pi r_\odot^2 n_* r}.$$

This expression is less than 1 always, and to first order reproduces our more naive result above,

$$P(\text{star in cylinder}) = 1 - (1 - \pi r_\odot^2 n_* r + \frac{1}{2}(\pi r_\odot^2 n_* r)^2) - ...$$

When $\pi r_\odot^2 n_* R_{max} = 1/2$, we make an error of $(1/2)^3$ by keeping only the first order term.

2: Newtonian cosmology. In Newtonian gravity, we have a cloud of galaxies expanding. The gravitational force is as if all the mass inside the sphere is located at the origin, and so

$$\dot{v} = \dot{H} r + H\dot{r} = (\dot{H} + H^2) r = -GM/r^2.$$

Note that $M = \rho(t) 4\pi/3 r^3(t)$ is constant for a given galaxy at distance $r(t)$. If $r(t) = a(t) r_0$, we have

$$(\dot{H} + H^2) = -G4\pi\rho(t)/3, \qquad H = \dot{(a)}/a,$$

which is independent of r_0. This is equivalent to

$$\frac{\ddot{a}}{a} = -4\pi G\rho(t)/3.$$

(This is the 2nd Friedmann equation!).

The total energy of a galaxy is

$$\kappa = \frac{1}{2} m H^2 r^2 - GMm/r,$$

and so we have that

$$K = \frac{2\kappa}{r_0^2} = m\left(H^2 a^2 - 8\pi G\rho_0/3a\right),$$

which is again independent of r_0. This can be rewritten

$$H^2 = \frac{8\pi G\rho}{3} + \frac{K}{ma^2}.$$

(This is the first Friedmann equation!)

We see that the critical values for H and ρ are

$$H = \sqrt{\frac{8\pi G\rho}{3}}, \qquad \rho_c(t) = \frac{3H^2(t)}{8\pi G}.$$

For $H = 70$km/s/Mpc, we get $\rho_c = 9.2 \times 10^{-27}$kg/m^3.

3: Curved space. We wish to find the area of a spherical shell in a cruved space, at distance s from the origin. In flat space $K = 0$ this is just

$$A_0 = 4\pi s^2.$$

In curved space, we use the equation for the distance

$$s(r) = \frac{a}{\sqrt{K}}\chi, \quad K > 0,$$
$$s(r) = \frac{a}{\sqrt{|K|}}\chi, \quad K < 0,$$

where χ is a function of r which will turn out to not matter (sin or sinh). Then we want to find

$$Ads = \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_{\chi(s)}^{\chi(s+ds)} d\chi \sin(h)^2 \chi \sin\theta.$$

We have used the inverted relation for $\chi(s)$ to specify the limits of the integral. Also, we allow for $K < 0$, $K > 0$ in the sin vs. sinh prescription Calculating and expanding in ds to first order, we get

$$
\begin{aligned}
A_{>} &= \frac{4\pi a^2}{K}\frac{1}{2}\left(1-\cos(2c)\right), \quad K>0, \\
A_{<} &= \frac{4\pi a^2}{K}\frac{1}{2}\left(\cosh(2c)-1\right), \quad K<0,
\end{aligned}
$$

c is here $s/r_c = \sqrt{K}s/a$, for the various examples in the question. These are

$$
\begin{aligned}
c &= 0.1, & \tfrac{A_>}{A_0} &= 0.9967, & \frac{A_<}{A_0} &= 1.0033, \\
c &= 1, & \tfrac{A_>}{A_0} &= 0.708, & \frac{A_<}{A_0} &= 1.381, \\
c &= 3, & \tfrac{A_>}{A_0} &= 0.0022, & \frac{A_<}{A_0} &= 11.15, \\
c &= 10, & \tfrac{A_>}{A_0} &= 0.0030, & \frac{A_<}{A_0} &= 1210000,
\end{aligned}
$$