Cosmology: Homework 3. Solutions.

1: Energy continuity. Differentiate the first Friedmann equation to find

$$2\dot{a}/a(\ddot{a}/a - (\dot{a}/a)^2) = 8\pi G/3\dot{\rho} + 2K\dot{a}/a^3.$$

Use the first Friedmann equation again to get rid of K, and the second Friedmann equation to get rid of $\ddot{a}$, and it all reduces to

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}.$$

But this also comes from thermodynamics. If ρ is the energy per volume $= E/V$, we have

$$d\rho = dE/V - dV/V\rho$$

Then use that for adiabatic expansion (constant p), $dE = -pdV$ and that $dV/V = 3a^2da$, and we have

$$d\rho = -3(\rho + p)\frac{da}{a},$$

which is the same thing. Total energy is not conserved for the "gas", but it decreases as if it was an expanding gas, in accordance with thermodynamics. Note that for "matter", pressure p is zero, *total energy* is conserved, but the *energy density* decreases as a result of cosmological expansion.

2: Conformal time. Simply do the substitution $dt = a(\eta)d\eta$, to find

$$ds^2 = a^2(\eta)\left(d\eta^2 + ...\right)$$

and for the Friedmann equations ($a' = da/d\eta$)

$$\left(\frac{a'}{a^2}\right)^2 = ..., \qquad \frac{a''}{a^2} - \frac{a'}{a^3} = ...$$

3: Matter dominated Universe. Matter domination, $p = 0$, $\rho \propto a^{-3}$. Spatially flat

$$H^2 = \frac{8\pi G}{3}\rho(t).$$

Easiest is to assume $a(t) = a_0(t/t_0)^\alpha$, for some α, to find

$$\alpha^2 = \frac{8\pi G}{3}\rho_0 t_0^{3\alpha}/a_0^3, \qquad -2 = -3\alpha.$$

One of the a_0, t_0 and ρ_0 are redundant, but we have

$$\rho_0 t_0^2/a_0^3 = \frac{1}{6\pi G}.$$

Plugging into the second Friedmann equation, we get the same. The continuity equation above determines $\rho \propto a^{-3}$, and that one also came directly from the Friedmann equations.