Cosmology: Homework 4. Solutions.

1: Flat Universe with vacuum energy. Assume that the cosmological parameters are

$$H_0 = 68.7 \text{ km/s/Mpc}, \quad \Omega_m = 0.273, \quad \Omega_\Lambda = 0.727,$$

with the other energy components zero.

We then need the integral

$$t = H_0^{-1} \int_0^1 \frac{dx}{\sqrt{\Omega_m x^{-1} + \Omega_\Lambda x^2}}.$$

Rewrite as

$$t = H_0^{-1} \frac{2}{3\sqrt{\Omega_\Lambda}} \int_0^{\sinh^{-1}(1/b)} dy = \frac{2}{3} \frac{H_0^{-1}}{\sqrt{0.727}} \sinh^{-1}\left(\frac{0.727}{0.273}\right) = 1.334 H_0^{-1} = 1.334 \times 0.687^{-}1 \times 9.78 \times 10^9 \, a = 19 \times 10^9 \, a,$$

using

$$b = \sqrt{\Omega_m/\Omega_\Lambda}, \qquad y = \sinh^{-1}(x^{3/2}/b).$$

Ω_Λ is constant in time, whereas $\Omega_m \propto a^{-3}$. Hence there was equality between matter and vacuum energy at redshift z with

$$\frac{\Omega_\Lambda}{\Omega_m} = (1+z)^3 \rightarrow z = 0.386.$$

This corresponds to an age of the Universe of 9.81×10^9 years (use the equation above with $\Omega_\Lambda = \Omega_m$).

We can redo the integral above with any upper limit $1/(1+z)$, to get

$$H_0 t = \frac{2}{3\sqrt{\Omega_\Lambda}} \sinh^{-1}((1+z)^{-3/2}/b).$$

We invert this to find

$$\frac{a(t)}{a_0} = \frac{1}{1+z} = \left[b \sinh\left(\frac{3\sqrt{\Omega_\Lambda}}{2} H_0 t \right) \right]^{2/3}.$$

The Universe started accelerating when $\ddot{a} > 0$, so when

$$t = \frac{1}{3\sqrt{\Omega_\Lambda} H_0} \cosh^{-1}(2) = 7.32 \times 10^9 a.$$

2: Evolution of the Hubble and density parameters. Assume a Universe with only non-relativistic matter ($\Omega_m = \Omega_0 \neq 0$, the others = 0). We have the Friedmann equation, slightly rewritten,

$$H^2 = H_0^2 \Omega_m (1+z)^3 - \frac{K}{a^2}.$$

Then we use that

$$\frac{K}{H_0^2 a_0^2} = (\Omega_0 - 1).$$

Since $(1+z) = a_0/a$, we immediately get

$$H^2 = H_0^2(1+z)^2(1+\Omega_0 z)),$$

which is the desired result.

Similarly, starting from

$$\Omega = 1 + \frac{K}{a^2H^2},$$

using again

$$\frac{K}{H_0^2 a_0^2} = (\Omega_0 - 1),$$

to get rid of K and the final result above to replace H^2, we find

$$\Omega = \frac{\Omega_0(1+z)}{1+\Omega_0 z}.$$