Cosmology: Homework 5. Solutions.

1: Age of the closed Universe. Similar as the other week, we need to calculate

$$t(z) = H_0^{-1} \int_0^{1/1+z} \frac{dx}{\sqrt{1 - \Omega_m + \Omega_m x^{-1}}},$$

for the case $\Omega_m > 1$, and in the limit $z \to 0$. Turns out it is easier in conformal time $dt = x d\eta$, so we have

$$\int d\eta = H_0^{-1} \int_0^{1/1+z} \frac{dx}{x\sqrt{1 - \Omega_m + \Omega_m x^{-1}}}.$$

We observe that we can write it as

$$\eta(z) = \int_0^{1/1+z} \frac{dx}{\sqrt{(\Omega_m - 1)x}\sqrt{(\frac{\Omega_m}{\Omega_m - 1} - x)}}.$$

Then do the substitution

$$x = \frac{\Omega_m}{\Omega_m - 1}\sin^2\phi, \quad dx = \frac{\Omega_m}{\Omega_m - 1}\sin\phi\cos\phi d\phi.$$

Then the integral is just

$$\eta(z) = \frac{2H_0^{-1}}{\sqrt{\Omega_m - 1}}\phi_f, \qquad \phi_f = \sin\left(\sqrt{\frac{\Omega_m - 1}{\Omega_m}\frac{1}{1+z}}\right).$$

Then we need to integrate this up to get t. First invert to get

$$\frac{1}{1+z}(\eta) = \frac{a}{a_0}(\eta) = x(\eta).$$

Then calculate ($\eta_f = \eta(\phi_f)$)

$$t = \int_0^{\eta_f} x d\eta = \frac{H_0^{-1}\Omega_m}{(\Omega_m - 1)^{3/2}}\left(2\sin^{-1}\sqrt{\frac{\Omega_m - 1}{\Omega_m}} - \sin\left(2\sin^{-1}\sqrt{\frac{\Omega_m - 1}{\Omega_m}}\right)\right).$$

We get (I think...)

$$\Omega_m = 1.1 \to t_{\rm age} = 0.688 H_0^{-1}, \quad \Omega_m = 2 \to t_{\rm age} = 0.571 H_0^{-1}.$$

Notice that the problem set had a typo: the Hubble rate should have been 70 rather than 10. If you used 10, you should get ages that are 7 times larger.

2: Gamma ray bursts. We have from the notes that the total emitted energy is

$$E_{tot} = 4\pi l d_L^2,$$

where l is either $l_{peak} = 2.7 \times 10^{-9} W/m^2$ or *fluence* $3.5 \times 10^{-7} J/m^2$.

We need to calculate d_L^2. In the flat case a) $\Omega_m = 0.273$, $\Omega_\Lambda = 0.727$, we read off that $d_L \simeq 2.8 H_0^{-1}$. After plugging everything in and converting units, we find

$$\text{Absolute luminosity} \quad 4.6 \times 10^{45} W,$$
$$\text{Total energy} \quad 6 \times 10^{47} J \simeq 3 m_\odot.$$

We can do the actual calculation in the simpler case $\Omega_m = 0.273$, $\Omega_\Lambda = 0.0$. We have

$$d_L = (1+z)H_0^(-1)\frac{1}{\sqrt{1-\Omega_m}}\sinh\left(\sqrt{1-\Omega_m}I\right),$$

with the integral

$$I = \int_{1/1+z}^{1}\frac{dx}{\sqrt{\Omega_m(x-x^2)+x^2}}.$$

This is almost identical to the integral above, except that now $1-\Omega_m > 0$ and so the substitution relevant is $x \propto \sinh^2\phi$ (instead of $\sin\phi$). The rest goes through the same, and we finally have

$$E_{tot} = 4\pi l\frac{2.6^2H_0^{-2}}{1-0.273}\sinh^2\left(2\sinh^{-1}\sqrt{\frac{1-\Omega_m}{\Omega(1+z)}} - \sinh^{-1}\sqrt{\frac{1-\Omega_m}{\Omega_m}}\right).$$

Using $h = 0.7$, we get

Absolute luminosity $3.7\times10^{45}W$,
Total energy $4.9\times10^{47}J \simeq 2.7m_\odot$.