Cosmology: Homework 6. Solutions.

1: Matter radiation equality. The current ratio of matter to radiation is

$$\frac{\rho_m^0}{\rho_r^0} = \frac{3M_{\rm pl}^2 H_0^2 \Omega_m}{\frac{\pi^2}{15}\left(1+(21/8)(4/11)^{1/3}\right)T_0^4}.$$

Putting in $H_0 = 70$km/sMpc and $T_0 = 2.725$K and converting units we get

$$1 + z_{eq} = 6833\Omega_m.$$

Since $\rho_r \propto a^{-4}$ and $\rho_r \propto a^{-3}$, this is also the growth factor of the Universe, since the time $t_{\rm eq}$ when the density components were equal. The temperature then was just

$$6833\Omega_m 2.725\text{K} = 1.6\text{eV}\Omega_m.$$

To find the age of the Universe t_eq, we take the age-redshift relation in radiation domination (before equality)

$$t(z) = H_0^{-1}\int_0^{1/6833}\frac{dx}{\sqrt{(1-\Omega_r)+\Omega_r x^{-2}}}.$$

Using $\Omega_r = \Omega_m/6833$, we get for $\Omega_m = 0.1, 0.3, 1.0$,

$$t_{\rm eq} = 39000\text{yrs}, \quad 22600\text{yrs}, \quad 12400\text{yrs}.$$

2: Thermal distributions in non-relativistic limit. In the non-relativistic limit,

$$\sqrt{p^2+m^2} - \mu \simeq m + \frac{p^2}{2m} - \mu,$$

and we have

$$n = \frac{g}{(2\pi)^3}\int \frac{d^3p}{e^{\frac{\sqrt{p^2+m^2}-\mu}{T}} \pm 1} \simeq \frac{g}{2\pi^2}e^{\frac{m-\mu}{T}}\int p^2 e^{-\frac{p^2}{mT}} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T},$$

where in the second equality sign we have also ignored the ±1 in the denominator. Similarly

$$\rho = \frac{g}{(2\pi)^3}\int \frac{\left(m+\frac{p^2}{2m}\right)d^3p}{e^{\frac{\sqrt{p^2+m^2}-\mu}{T}} \pm 1} = mn + \frac{g}{4\pi^2 m}e^{\frac{m-\mu}{T}}\int p^4 e^{-\frac{p^2}{mT}} = mn + \frac{3T}{2}n.$$

Also

$$P = \frac{g}{(2\pi)^3}\int \frac{p^2}{3E}\frac{d^3p}{e^{\frac{\sqrt{p^2+m^2}-\mu}{T}} \pm 1} \simeq \frac{g}{6m\pi^2}e^{\frac{m-\mu}{T}}\int p^4 e^{-\frac{p^2}{mT}} = nT.$$

It immediately follows that

$$\langle E\rangle = \frac{\rho}{n} = m + \frac{3T}{2},$$

and

$$n - \bar{n} = n(\mu) - n(-\mu) = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} 2\sinh\frac{\mu}{T}.$$