Cosmology: Homework 7. Solutions.

1: Tremaine-Gunn limit The phase space of the neutrino gas is limited by a ball in position space and a ball in momentum space

$$V_x = \frac{4\pi}{3} r^3, \qquad V_p = m_\nu^3 \frac{4\pi}{3} v_{\rm esc}^3.$$

The escape and rotational velocities are given in terms of the mass

$$v_{\rm esc} = \frac{2GM}{r}, \qquad v_{\rm rot}^2 = \frac{GM}{r}.$$

The number of states per unit phase space volume is $g/(2\pi\hbar)^3$, so we have than the total mass is

$$M = \frac{r v_{\rm rot}^2}{G} = m_\nu^4 \frac{g}{(2\pi\hbar)^3} \left(\frac{4\pi}{3}\right)^2 r^3 v_{\rm esc}^3.$$

Using the proper conversion factors, and $r = 10$kpc, $v_{\rm rot} = 220$km/s, we find

$$m_\nu = \frac{53\text{eV}}{g},$$

which is 26.5eV with one heavy species ($g = 2$) and $\simeq 9$eV for three lighter species.

2: Redshift of non-relativistic particles

Consider the equilibrium distribution, for non-relativistic particles

$$n_p = \frac{1}{\exp\left(\frac{\sqrt{p^2+m^2}-\mu}{T}\right) \pm 1} \simeq \frac{1}{\exp\left(\frac{p^2/(2m)+m-\mu}{T}\right) \pm 1}$$

p redshifts as $p_2 = (a_1/a_2)p_1$. We wish to keep the form of the distribution (the scale factors multiplying n_p cancel out, as for the relativistic case $n_p \to a^3/a^3 n_p = n_p$, so we only need to check the exponent, and ensure that by a proper choice of T_2 and μ_2, we have

$$\frac{\frac{p_1^2}{2} + m - \mu_1}{T_1} = \frac{\frac{p_2^2}{2m} + m - \mu 2}{T_2}.$$

The first term tells us that

$$T_2 = \left(\frac{a_1}{a_2}\right)^2 T_1,$$

temperature decreases as for the relativistic case, but now with the *square* of the scale factor. This gives for the chemical potential

$$\mu_2 = \mu_1 \left(\frac{a_1}{a_2}\right)^2 + m\left(1 - \left(\frac{a_1}{a_2}\right)^2\right),$$

which means that over time ($a_1/a_2 \to 0$), the original μ_1 gets "washed out", and the new chemical potential goes asymptotically to $\mu_2 = m$.