1. **Olbers' paradox**.

   (a) Let's assume the universe is infinite, eternal, and unchanging (and has Euclidean geometry). For simplicity, lets also assume that all stars are the same size as the sun, and distributed evenly in space. Show that the line of sight meets the surface of a star in every direction, sooner or later. Use Euclidean geometry.

   (b) Let's put in some numbers: the luminosity density of the Universe is $10^8 L_\odot/\text{Mpc}^3$ (within a factor of 2). With the above assumption we have then a number density of stars $n_\star = 10^8\text{Mpc}^{-3}$. The radius of the sun is $r_\odot = 7 \times 10^8 m$. Define $r_{1/2}$ so that stars closer than $r_{1/2}$ cover 50% of the sky. Calculate $r_{1/2}$.

   (c) Let's assume instead that stars have finite ages: they all appeared $t = 4.6 \times 10^9$a ago. What fraction f of the sky do they now cover? What is the energy density of starlight in the universe, in kg/m^3? (The luminosity, or radiated power, of the sun is $L_\odot = 3.85 \times 10^{26}$W).

   (d) Calculate $r_{1/2}$ and f for galaxies, using $n_G = 3 \times 10^{-3}\text{Mpc}^{-3}$, $r_G = 10$kpc, and $t_G = 10^{10}$a.

2. **Newtonian cosmology**. Use Euclidean geometry and Newtonian gravity, so that we interpret the expansion of the universe as an actual motion of galaxies instead of an expansion of space itself. Consider thus a spherical group of galaxies in otherwise empty space. At a sufficiently large scale you can treat this as a homogeneous cloud (the galaxies are the cloud particles). Let the mass density of the cloud be $\rho(t)$. Assume that each galaxy moves according to Hubbles law $v(t, \mathbf{r}) = H(t)\mathbf{r}$. The expansion of the cloud slows down due to its own gravity. What is the acceleration as a function of ρ and $r \equiv |r|$? Express this as an equation for $\dot{H}(t)$. Choose some reference time $t = t_0$ and define $a(t)r(t)/r(t_0)$. Show that $a(t)$ is the same function for each galaxy, regardless of the value of $r(t_0)$. Note that $\rho(t) = \rho(t_0)a(t)^3$. Rewrite your differential equation for $H(t)$ as a differential equation for $a(t)$. You can solve $H(t)$ also using energy conservation. Denote the total energy (kinetic + potential) of a galaxy per unit mass by κ. Show that $K \equiv 2\kappa/r(t_0)^2$ has the same value for each galaxy, regardless of the value of $r(t_0)$. Relate $H(t)$ to $\rho(t_0)$, K, and $a(t)$. Whether the expansion continues forever, or stops and turns into a collapse, depends on how large H is in relation to ρ. Find out the critical value for H (corresponding to the escape velocity for the galaxies) separating these two possibilities. Turn the relation around to give the *critical density* corresponding to a given "Hubble constant" H. What is this critical density (in kg/m^3) for $H = 70$ km/s/Mpc?

3. **Curved space.** Consider the spatial part of the Robertson-Walker metric (at time t when $a(t) = a$) in the cases $K > 0$ and $K < 0$. What is the volume of the spherical shell whose distance from the origin is between s and $s + ds$? What is the deviation from the Euclidean ($K = 0$) result when $s = 0.1r_{\text{curv}}$, $s = r_{\text{curv}}$, $s = 3r_{\text{curv}}$, and $s = 10r_{\text{curv}}$? ($r_{\text{curv}} = a/\sqrt{|K|}$).