1. For plane or cylindrical polar coordinates $\hat{\mathbf{r}} = \boldsymbol{i}\cos\theta + \boldsymbol{j}\sin\theta$ and $\hat{\boldsymbol{\theta}} = -\boldsymbol{i}\sin\theta + \boldsymbol{j}\cos\theta$, see appendix B of the lectures. Express $\mathbf{i}$, $\mathbf{j}$ in terms of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$.
   Solution:
   We start from $\boldsymbol{r} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$. From vector analysis, we know that (see the appendix of the lecture notes)
   $$\hat{\mathbf{r}} = \frac{\partial\mathbf{r}/\partial\mathbf{r}}{|\partial\mathbf{r}/\partial\mathbf{r}|} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j},$$
   and
   $$\hat{\boldsymbol{\theta}} = \frac{\partial\mathbf{r}/\partial\theta}{|\partial\mathbf{r}/\partial\theta|} = \frac{-r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j}}{r} = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}.$$
   Multiplying the first equation by $\sin\theta$, the second by $\cos\theta$ and adding both sides gives $\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}} = (\sin^2\theta + \cos^2\theta)\mathbf{j} = \mathbf{j}$.
   On the other hand, multiplying the first equation by $\cos\theta$, the second by $-\sin\theta$ and adding both sides gives $\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} = (\cos^2\theta + \sin^2\theta)\mathbf{i} = \mathbf{i}$, and thus we get finally
   $$\mathbf{i} = \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}} \qquad\qquad \mathbf{j} = \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}}.$$

2. Generalise the one dimensional Taylor's theorem for three dimensions $\phi(x_1 + h_1, x_2 + h_2, x_3 + h_3)$ by considering all the coordinates separately and ending at second degree terms in $\mathbf{h}$. Show that it may be put as
   $$\phi(\mathbf{x} + \mathbf{h}) = \phi(\mathbf{x}) + \mathbf{h}\cdot(\nabla\phi)_{\mathbf{x}} + O(\mathbf{h}^2) = \phi(\mathbf{x}) + h_j(\partial\phi/\partial x_j)_{\mathbf{x}} + O(h_k h_k). \qquad (1)$$
   Solution:
   Using Taylor's theorem one can approximate the function $f(x)$ around point x_0 by
   $$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + O(x^3).$$
   We can use the theorem to approximate $f(x+h)$ near point x: $f(x+h) = f(x) + f'(x)h + O(h^2)$. This is now the Taylor's theorem in 1D that we need.
   In three dimensions, we apply the theorem to each coordinate separately and start from coordinate x_i.
   $$\phi(\mathbf{x} + h_i\hat{\mathbf{x}}_i) = \phi(\mathbf{x}) + \left(\frac{\partial\phi}{\partial x_i}\right)_{\mathbf{x}} h_i + O(h_i^2). \qquad (2)$$
   We then approximate this around point $\mathbf{x} + h_i\hat{\mathbf{x}}_i$ with respect to another coordinate x_j, $i \neq j$, by
   $$\phi(\mathbf{x} + h_i\hat{\mathbf{x}}_i + h_j\hat{\mathbf{x}}_j) = \phi(\mathbf{x} + h_i\hat{\mathbf{x}}_i) + \left(\frac{\partial\phi}{\partial x_j}\right)_{\mathbf{x}+h_i\hat{\mathbf{x}}_i} h_j + O(h_j^2).$$

The above partial derivative is approximated as in Eq. (**??**) and can be written as

$$\left(\frac{\partial\phi}{\partial x_j}\right)_{\mathbf{x}+h_i\hat{\mathbf{x}}_i} = \left(\frac{\partial\phi}{\partial x_j}\right)_{\mathbf{x}} + h_i\left(\frac{\partial^2\phi}{\partial x_j\partial x_i}\right)_{\mathbf{x}} + O(h^2),$$

and using again Eq. (**??**) for $\phi(\mathbf{x}+h_i\hat{\mathbf{x}}_i)$ we end up with

$$\phi(\mathbf{x}+h_i\hat{\mathbf{x}}_i+h_j\hat{\mathbf{x}}_j) = \phi(\mathbf{x}) + \left(\frac{\partial\phi}{\partial x_i}\right)_{\mathbf{x}} h_i + \left(\frac{\partial\phi}{\partial x_j}\right)_{\mathbf{x}} h_j + O(h^2),$$

where h^2 includes h_i^2, h_j^2 and h_ih_j. The third coordinate is treated similarly, and we get

$$\phi(\mathbf{x}+\mathbf{h}) = \phi(\mathbf{x}) + \sum_{i=1}^{3}\left(\frac{\partial\phi}{\partial x_i}\right)_{\mathbf{x}} h_i + O(h^2) = \phi(\mathbf{x}) + \mathbf{h}\cdot(\nabla\phi)_{\mathbf{x}} + O(h^2).$$

3. For spherical polar coordinates calculate $\partial\hat{\mathbf{r}}/\partial\theta$, $\partial\hat{\boldsymbol{\theta}}/\partial\theta$, $\partial\hat{\boldsymbol{\theta}}/\partial\lambda$, $\partial\hat{\boldsymbol{\lambda}}/\partial\lambda$; in each case express your answer in terms of the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\lambda}}$ and not in terms of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.
   Solution: For notational beauty and shortness, we use now notation: $\partial_x f = \frac{\partial f}{\partial x}$.
   The position vector in spherical polar coordinates is (see the appendix of the lecture notes)

$$\boldsymbol{r} = r\sin\theta\cos\lambda\boldsymbol{i} + r\sin\theta\sin\lambda\boldsymbol{i} + r\cos\theta\boldsymbol{k}$$

   The definition of the unit vectors

$$\hat{\mathbf{r}} = \frac{\partial_r\boldsymbol{r}}{|\partial_r\boldsymbol{r}|} = \frac{\sin\theta\cos\lambda\boldsymbol{i} + \sin\theta\sin\lambda\boldsymbol{i} + \cos\theta\boldsymbol{k}}{1} = \sin\theta\cos\lambda\boldsymbol{i} + \sin\theta\sin\lambda\boldsymbol{i} + \cos\theta\boldsymbol{k}$$

$$\hat{\boldsymbol{\theta}} = \frac{\partial_\theta\boldsymbol{r}}{|\partial_\theta\boldsymbol{r}|} = \frac{r\cos\theta\cos\lambda\boldsymbol{i} + r\cos\theta\sin\lambda\boldsymbol{i} - r\sin\theta\boldsymbol{k}}{r} = \cos\theta\cos\lambda\boldsymbol{i} + \cos\theta\sin\lambda\boldsymbol{i} - \sin\theta\boldsymbol{k}$$

$$\hat{\boldsymbol{\lambda}} = \frac{\partial_\lambda\boldsymbol{r}}{|\partial_\lambda\boldsymbol{r}|} = \frac{-r\sin\theta\sin\lambda\boldsymbol{i} + r\sin\theta\cos\lambda\boldsymbol{j}}{r\sin\theta} = -\sin\lambda\boldsymbol{i} + \cos\lambda\boldsymbol{j}$$

   The partial derivatives $\partial_\theta\hat{\mathbf{r}}$, $\partial_\theta\hat{\boldsymbol{\theta}}$, $\partial_\lambda\hat{\boldsymbol{\theta}}$ and $\partial_\lambda\hat{\boldsymbol{\lambda}}$ are calculated in the above representation since the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ do not depend on the variables r, θ, λ.

$$\begin{aligned}
\partial_\theta\hat{\mathbf{r}} &= \cos\theta\cos\lambda\boldsymbol{i} + \cos\theta\sin\lambda\boldsymbol{j} - \sin\theta\boldsymbol{k} = \hat{\boldsymbol{\theta}} \\
\partial_\theta\hat{\boldsymbol{\theta}} &= -\sin\theta\cos\lambda\boldsymbol{i} - \sin\theta\sin\lambda\boldsymbol{j} - \cos\theta\boldsymbol{k} = -\hat{\mathbf{r}} \\
\partial_\lambda\hat{\boldsymbol{\theta}} &= -\cos\theta\sin\lambda\boldsymbol{i} + \cos\theta\cos\lambda\boldsymbol{j} = \cos\theta\hat{\boldsymbol{\lambda}} \\
\partial_\lambda\hat{\boldsymbol{\lambda}} &= -(\cos\lambda\boldsymbol{i} + \sin\lambda\boldsymbol{j}) = -(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\boldsymbol{\theta}})
\end{aligned}$$

4. The following identities and notations are extensively used in this exercise, and remember

the Einstein summation rule $\sum_{i=1}^{3} a_i b_i = a_i b_i$.

$$\boldsymbol{A} \times \boldsymbol{B} = (\hat{\mathbf{e}}_n A_n) \times (\hat{\mathbf{e}}_m B_m) = \epsilon_{ijk} \hat{\mathbf{e}}_i A_j B_k \qquad \boldsymbol{A} \cdot \boldsymbol{B} = (\hat{\mathbf{e}}_n A_n) \cdot (\hat{\mathbf{e}}_m B_m) = A_i B_i$$

$$\nabla = \hat{\mathbf{e}}_i \partial_i \qquad \nabla \phi = \hat{\mathbf{e}}_i \partial_i \phi$$

$$\nabla \cdot \boldsymbol{A} = \partial_i A_i \qquad \nabla \times \boldsymbol{A} = \epsilon_{ijk} \hat{\mathbf{e}}_i \partial_j A_k$$

$$\nabla^2 = \sum_{i=1}^{3} \partial_i^2 = \partial_i \partial_i = \partial_{ii}$$

One should also recall that derivation is a linear operation: $\partial_j(A+B+C) = \partial_j A + \partial_j B + \partial_j C$ and the derivative of the product $\partial_j(AB) = \partial_j A + \partial_j B$.

Solutions:

(a) If vector field $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}, t)$ is well behaved in the sense of derivation then the order of derivatives is interchangeable. Notation of $\partial_{jt} = \frac{\partial^2}{\partial j \partial t}$ is in addition applied.

$$\nabla \times (\partial_t \boldsymbol{A}) = \epsilon_{ijk} \hat{\mathbf{e}}_i \partial_{jt} A_k = \partial_t (\epsilon_{ijk} \hat{\mathbf{e}}_i \partial_j A_k) = \partial_t (\nabla \times \boldsymbol{A})$$

(b) By applying the product rule of derivation:

$$\begin{aligned} \nabla \cdot (\phi \nabla \psi) &= (\hat{\mathbf{e}}_n \partial_n) \cdot (\phi (\hat{\mathbf{e}}_m \partial_m \psi)) = (\hat{\mathbf{e}}_n \partial_n) \cdot (\hat{\mathbf{e}}_m (\phi \partial_m \psi)) \\ &= \underbrace{\partial_k (\phi \partial_k \psi)}_{\text{product rule}} \\ &= \partial_k \phi \partial_k \psi + \phi \partial_{kk} \psi \\ &= (\nabla \phi) \cdot (\nabla \psi) + \phi \nabla^2 \psi \end{aligned}$$

(c) The vector field $\boldsymbol{A} = \boldsymbol{A}(r, t)$ is assumed well behaved that the interchange of the order of the partial derivation holds: $\partial_{jk} A_l = \partial_{kj} A_l$.

$$\begin{aligned} \nabla \cdot (\nabla \times \boldsymbol{A}) &= (\hat{\mathbf{e}}_n \partial_n) \cdot (\epsilon_{ijk} \hat{\mathbf{e}}_i \partial_j A_k) = \epsilon_{ijk} \partial_{ij} A_k \\ &= \partial_{12} A_3 + \partial_{31} A_2 + \partial_{23} A_1 - \partial_{13} A_2 - \partial_{32} A_1 - \partial_{21} A_3 \\ &= \underbrace{(\partial_{12} A_3 - \partial_{21} A_3)}_{=0} + \underbrace{(\partial_{31} A_2 - \partial_{13} A_2)}_{=0} + \underbrace{(\partial_{23} A_1 - \partial_{32} A_1)}_{=0} = 0 \end{aligned}$$

At the second row, the permutation symbol ϵ_{ijk} is explicitly expressed (see the appendix of the lecture notes).

(d)

$$\begin{aligned} \nabla \times (\nabla \phi) &= (\hat{\mathbf{e}}_n \partial_n) \times (\hat{\mathbf{e}}_m \partial_m \phi) = \epsilon_{ijk} \hat{\mathbf{e}}_i \partial_{jk} \phi \\ &= \hat{\mathbf{e}}_1 \underbrace{(\partial_{23} \phi - \partial_{32} \phi)}_{=0} + \hat{\mathbf{e}}_2 \underbrace{(\partial_{31} \phi - \partial_{13} \phi)}_{=0} + \hat{\mathbf{e}}_3 \underbrace{(\partial_{12} \phi - \partial_{21} \phi)}_{=0} = 0 \end{aligned}$$

(d)

$$\begin{aligned} \nabla \times (\nabla \times \boldsymbol{A}) &= \epsilon_{ijk} \hat{\mathbf{e}}_i \partial_j (\nabla \times A)_k = \epsilon_{ijk} \hat{\mathbf{e}}_i \partial_j \epsilon_{klm} \partial_l A_m \\ &= \hat{\mathbf{e}}_i \epsilon_{ijk} \epsilon_{klm} \partial_{jl} A_m = -\hat{\mathbf{e}}_i \epsilon_{ijk} \epsilon_{mlk} \partial_{jl} A_m \\ &= -\hat{\mathbf{e}}_i (\delta_{im} \delta_{jl} - \delta_{il} \delta_{jm}) \partial_{jl} A_m = -\hat{\mathbf{e}}_i \partial_{jj} A_i + \hat{\mathbf{e}}_i \partial_{ji} A_j \\ &= (\hat{\mathbf{e}}_i \partial_i)(\partial_j A_j) - \nabla^2 \boldsymbol{A} = \nabla (\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A} \end{aligned}$$

For $\epsilon_{ijk}\epsilon_{mlk}$, it has been applied the identity derived in the appendix of the lectures.

5. An estimate for the mean free path λ of gas particles can be based on the equation

$$\lambda \approx \frac{1}{n\sigma}, \tag{3}$$

where $n = N/V$ is the number density of particles and σ is the scattering cross section. Estimate λ in standard temperature and pressure ($T = 273$ K and $p = 10^5$ Pa) using this formula and $\sigma \approx \pi a^2$, where $a = 150$ pm is the bond length of N_2 molecule. Use the equation of state of an ideal gas $pV = Nk_BT$, where the Boltzmann's constant $k_B = 1.381 \cdot 10^{-23}$ J K^{-1}.

(Note that equation (**??**) can be derived as follows: when a particle of cross section σ travels distance λ, it sweeps volume $V = \sigma\lambda$. For one collision to occur in this distance, we must have approximately one particle in this volume, $nV \approx 1$.)
Solution: The number density n of an ideal gas is

$$n = \frac{N}{V} = \frac{P}{k_{\mathrm{B}}T}$$

and the estimate for the scattering cross section for N_2 molecule $\sigma = \pi a^2$. Plugging these formulas and given values for P, T and a in the formula of the guestimated mean free path λ of a particle in an N_2 ideal gas:

$$\lambda \approx \frac{1}{n\sigma} = \frac{k_{\mathrm{B}}T}{P\pi a^2} = \frac{1.381 \cdot 10^{-23}\,\mathrm{NmK}^{-1}\,273\,\mathrm{K}}{10^5\,\mathrm{Nm}^{-2}\pi(150)^2 10^{-24}\,\mathrm{m}^2} = 0.533\,\mu\mathrm{m} \approx 0.5\mu\mathrm{m}$$