1. Show that

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot\, d\boldsymbol{S}.$$

Solution: In the previous exercise (4.b), it was shown that

$$\nabla \cdot (\phi\nabla\psi) = (\nabla\phi) \cdot (\nabla\psi) + \phi\nabla^2\psi$$

By reordering this identity:

$$\begin{aligned}
&\phi\nabla^2\psi = \nabla \cdot (\phi\nabla\psi) - (\nabla\phi) \cdot (\nabla\psi) \quad \text{by interchanging } \phi \leftrightarrow \psi \\
&\psi\nabla^2\phi = \nabla \cdot (\psi\nabla\phi) - (\nabla\psi) \cdot (\nabla\phi) \quad \text{difference of these equations} \\
&\phi\nabla^2\psi - \psi\nabla^2\phi = \nabla \cdot (\phi\nabla\psi) - \nabla \cdot (\psi\nabla\phi) = \nabla \cdot (\phi\nabla\psi - \psi\nabla\phi) \qquad (1)
\end{aligned}$$

In course of Mathematics for physics, it was introduced the divergence theorem or Gauss' theorem:

$$\int_V \nabla \cdot \boldsymbol{F}\, dV = \int_S \boldsymbol{F} \cdot\, d\boldsymbol{S}, \quad \text{where S is the boundary of the volume V.} \qquad (2)$$

Combining equations (1) and (2) produces

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV \overset{(1)}{=} \int_V \nabla \cdot (\phi\nabla\psi - \psi\nabla\phi)\, dV \overset{(2)}{=} \int_S (\phi\nabla\psi - \psi\nabla\phi) \cdot d\boldsymbol{S}.$$

2. Calculate and describe particle paths and streamlines for the flow

$$\mathbf{v} = (ay, -ax, b(t)) \qquad (3)$$

What could be modelled by the case b(t)=constant?
Solution:
Particle paths Notation: $v_x = \frac{dx}{dt} = \dot{x}$, $v_y = \frac{dy}{dt} = \dot{y}$ and $v_z = \frac{dz}{dt} = \dot{z}$. The velocity in the component form is

$$\dot{x} = ay, \quad \dot{y} = -ax, \quad \dot{z} = b(t).$$

The x and y components are connected but the z component depends only on the function $b(t)$:

$$dotz = b(t) \;\Rightarrow\; dz = b(t)\, dt \;\Rightarrow\; \int_{z_0}^{z} dz = \int_0^t b(\tau)\, d\tau \;\Rightarrow\; z(t) = z_0 + \int_0^t b(\tau)\, d\tau.$$

For the x and y, the trick of additional derivation works well:

$$\dot{x} = ay \text{ time derivative on both sides} \Rightarrow \quad \ddot{x} = a\dot{y} = a(-ax) \Rightarrow \quad \ddot{x} + a^2 x = 0.$$

The last of the above equations is a standard differential equation, which has general solution of

$$x(t) = A\cos at + B\sin at$$

From the eqn. $y = \frac{\dot{x}}{a}$, the y component is

$$y(t) = -A\sin at + B\cos at$$

Coefficients A and B are solved from initial values $(x(0), y(0)) = (x_0, y_0)$, and the general particle paths of the flow (3) are

$$x(t) = x_0\cos at + y_0\sin at \qquad y(t) = -x_0\sin at + y_0\cos at \qquad z(t) = z_0 + \int_0^t b(\tau)\, d\tau. \qquad (4)$$

In the xy-plane, the particle paths are origo-centered circles of radius $\sqrt{x_0^2 + y_0^2}$. The drift in z direction from the initial point z_0 is determined through the time integral $\int_0^t b(\tau)\, d\tau$. If $b(t) = c$ then there is a constant drift in z direction and a particle path is a helix.

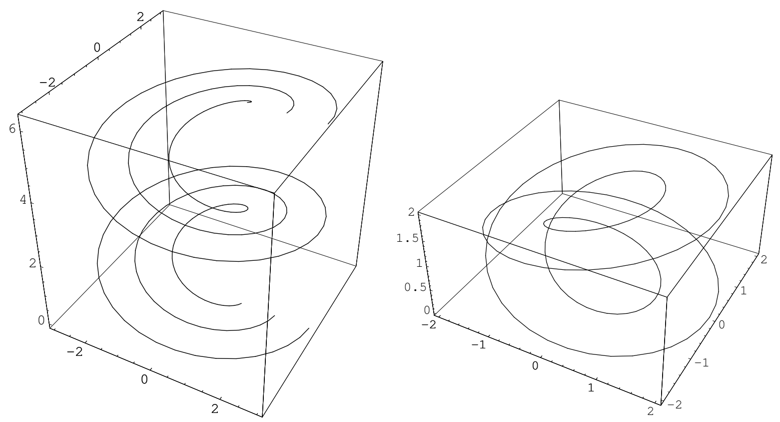

Figure 1: The particle paths of the equation group (4) with parameters (left figure): $b(t) = 1$, $z_0 = 0$, $a = 2$, $x_0 = 1, 2, 3$, $y_0 = 0$ and $t \in [0, 2\pi]$, (right figure): $b(t) = \sin t$, $z_0 = 0$, $a = 2$, $x_0 = 1, 2$, $y_0 = 0$ and $t \in [0, 2\pi]$.

Streamlines Notation: $\boldsymbol{p}(s) = (x(s), y(s), z(s))$ where the s is the arbitrary parametrization of the streamline $\boldsymbol{p}$. Now from the lectures: the definition of the streamline

$$\frac{d\boldsymbol{p}}{ds} = \boldsymbol{v}(\boldsymbol{p}(s), t)$$

$$\frac{dx(s)}{ds} = ay(s), \qquad \frac{dy(s)}{ds} = -ax(s), \qquad \frac{dz(s)}{ds} = b(t).$$

The solution of the x and y components is identical with the case of particle paths, but now $b(t)$ is constant with respect to the parameter s, and thus the streamlines are

$$x(s) = x_0 \cos as + y_0 \sin as \qquad y(s) = -x_0 \sin as + y_0 \cos as \qquad z(s) = z_0 + b(t)s. \tag{5}$$

Streamlines are at *any time* helices as in figure 1 (left). We have now demonstrated the fact that the particle paths of the time-dependent velocity field $\boldsymbol{v}(t)$ are not the same as streamlines. For example when $v_z(t) = b(t) = \sin t$, as in figure 1 (right), the particle paths are closed curves but the streamlines are open helices.

3. Sketch streamlines for

$$(a) \quad \mathbf{v} = (a\cos\omega t, a\sin\omega t, 0),$$
$$(b) \quad \mathbf{v} = (x - Vt, y, 0),$$
$$(c) \quad v_r = r\cos\frac{\theta}{2},\ v_\theta = r\sin\frac{\theta}{2},\ v_z = 0,\ 0 < \theta < 2\pi.$$

Solution:

(a) $\mathbf{v} = (a\cos\omega t, a\sin\omega t, 0)$

$$\frac{dx(s)}{ds} = a\cos\omega t \qquad \frac{dy(s)}{ds} = a\sin\omega t \qquad \frac{dz(s)}{ds} = 0$$
$$x(s) = x_0 + sa\cos\omega t \qquad y(s) = y_0 + sa\sin\omega t \qquad z(s) = z_0 \tag{6}$$

Evidently streamlines are constrained to the xy-plane at the z_0-altitude. The parametrization s is purely arbitrary, let's try to eliminate it to express streamlines in more concrete form

$$\begin{cases} y - y_0 = sa\sin\omega t \\ x - x_0 = sa\cos\omega t \end{cases} \quad \Rightarrow \quad \frac{y - y_0}{x - x_0} = \tan\omega t \tag{7}$$

Where x_0, y_0, z_0 are the coordinates of the streamline at parametrization point $s = 0$. The latter equation describe straight lines with time dependent slope $\tan\omega t$, see Fig. 2

(b) $\mathbf{v} = (x - Vt, y, 0)$

$$\frac{dx(s)}{ds} = x(s) - Vt \qquad \frac{dy(s)}{ds} = y(s) \qquad \frac{dz(s)}{ds} = 0$$
$$\pm x(s) = Ae^s + Vt \qquad y(s) = Be^s \qquad z(s) = z_0$$
$$\pm x(s) = (x_0 - Vt)e^s + Vt \qquad y(s) = \pm y_0 e^s \qquad z(s) = z_0$$

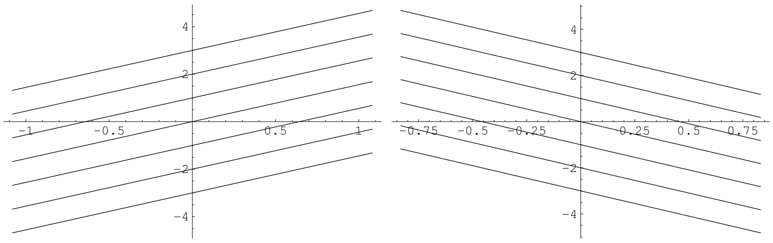

Figure 2: The streamlines of the equation group (7) with parameters (left figure): $t = 1$, $z_0 = 0$, $a = 1$, $x_0 = 0$, $y_0 = -3, -2, -1, 0, 1, 2, 3$ and $s \in [-2, 2]$, (right figure): same as left figure put $t = 2$.

The streamlines are now represented with parametrization $s \in [-\infty, \infty]$, let's make new parametrization $r = \pm e^s \in [-\infty, \infty]$.

$$x(s) = (x_0 - Vt)r + Vt \qquad y(s) = y_0 r \quad \text{are combined as}$$

$$x = \frac{x_0 - Vt}{y_0} y + Vt \tag{8}$$

which represents a straigth line $x = x(y)$ with time-dependent slope $(x_0 - Vt)/y_0$. The crossing point ($x = Vt$) of the x-axis travels with time to right with speed V and lines rotate counterclockwise as the slope $(x_0 - Vt)/y_0$ decreases with time, see Fig. 3.

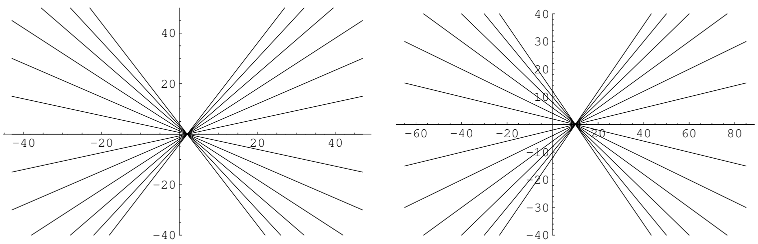

Figure 3: The streamlines of the equation group (8) with parameters (left figure): $t = 1$, $z_0 = 0$, $V = 3$, $x_0 = 5$, $y_0 = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$ and $r \in [0, 15]$; (right figure): same as left figure put $t = 5$.

(c) This is a bit more tricky exercies than the previous ones. $v_r = r\cos\frac{\theta}{2}$, $v_\theta = r\sin\frac{\theta}{2}$, $v_z = 0$, $0 < \theta < 2\pi$ First, one should know what is $\boldsymbol{v}$ in cylindrical polar coordinates:

$$\begin{aligned}\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \frac{d}{dt}(r\hat{\mathbf{r}} + z\boldsymbol{k}) = \dot{r}\hat{\mathbf{r}} + r\frac{d\mathbf{r}(\hat{\boldsymbol{\theta}})}{dt} + \dot{z}\boldsymbol{k} &= \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} + \dot{z}\boldsymbol{k} \\ &= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} + \dot{z}\boldsymbol{k} \\ &= v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}} + v_z\boldsymbol{k}\end{aligned}$$

Now, it is easy to identify relations $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$ and $v_z = \dot{z}$. Transforming these to s-parametrization:

$$\frac{dr(s)}{ds} = r\cos\frac{\theta}{2} \qquad r\frac{d\theta(s)}{ds} = r\sin\frac{\theta}{2} \qquad \frac{dz(s)}{ds} = 0$$
$$\frac{dr(s)}{ds} = r\cos\frac{\theta}{2} \qquad \frac{d\theta(s)}{ds} = \sin\frac{\theta}{2} \qquad z(s) = z_0$$

To sketch the streamlines in polar coordinates, our idea is to express the variable r as function of θ as we did with the previous streamlines, where y was expressed as a function of x. Let's study

$$\frac{dr}{d\theta} = \frac{dr/ds}{d\theta/ds} = \frac{r\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} \qquad \theta \neq 0 \qquad \Rightarrow$$
$$\frac{dr}{r} = 2\frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}d\theta \Rightarrow$$
$$\int\frac{dr}{r} = 2\int\frac{\frac{1}{2}d\theta\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$$

Remembering formula $\int\frac{f'}{f} = \ln|f| + C$ we find out

$$\ln|r| = 2\ln\sin\frac{\theta}{2} + C \Rightarrow \qquad r = r_0\sin^2\frac{\theta}{2} \tag{9}$$

where $r_0 = r(\pi)$.

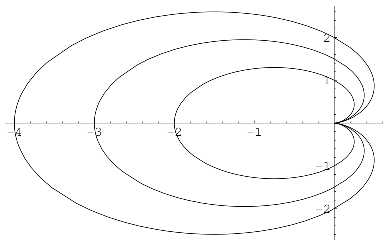

Figure 4: The streamlines of the equation (9) with parameters: $r_0 = 2, 3, 4$, $z_0 = 0$, and $\theta \in [0, 2\pi]$

4. Find streamlines and particle paths for the two-dimensional flows

$$(a) \quad \mathbf{v} = (xt, -yt, 0),$$
$$(b) \quad \mathbf{v} = (xt, -y, 0).$$

Solution: (a) $\mathbf{v} = (xt, -yt, 0)$
Solution procedure goes simoultanous in three columns

Particle paths

$$
\begin{aligned}
\dot{x} &= xt & \dot{y} &= -yt & \dot{z} &= 0 \\
\frac{dx}{x} &= tdt & \frac{dx}{y} &= -tdt & z(t) &= z_0 \\
\ln|x| &= \frac{1}{2}t^2 + C & \ln|y| &= -\frac{1}{2}t^2 + d & z(t) &= z_0 \\
x(t) &= \pm x_0 \mathrm{e}^{\frac{1}{2}t^2} & y(t) &= \pm y_0 \mathrm{e}^{-\frac{1}{2}t^2} & z(t) &= z_0 \\
xy &= \pm x_0 y_0 \Rightarrow & y &= \pm\frac{x_0 y_0}{x} & z &= z_0
\end{aligned}
\tag{10}
$$

Streamlines

$$
\begin{aligned}
\frac{dx}{ds} &= xt & \frac{dy}{ds} &= -yt & \frac{dz}{ds} &= 0 \\
x(s) &= \pm x_0 \mathrm{e}^{st} & y(s) &= y_0 \mathrm{e}^{-st} & z(s) &= z_0 \\
xy &= \pm x_0 y_0 \Rightarrow & y &= \pm\frac{x_0 y_0}{x} & z &= z_0
\end{aligned}
\tag{11}
$$

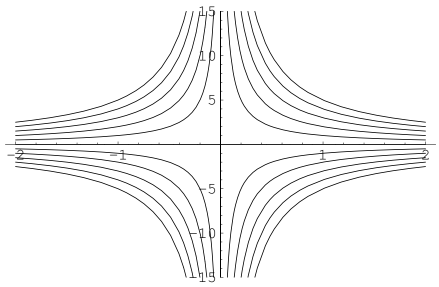

Figure 5: Particle paths (10) and streamlines (11) with intital values $x_0 = 1$ and $y_0 = -5, -4, -3, -2. - 1, 0, 1, 2, 3, 4, 5$

(b) $\mathbf{v} = (xt, -y, 0)$
Particle paths

$$
\begin{aligned}
\dot{x} &= xt & \dot{y} &= -y & \dot{z} &= 0 \\
\frac{dx}{x} &= tdt & \frac{dx}{y} &= -dt & z(t) &= z_0 \\
x(t) &= \pm x_0 \mathrm{e}^{\frac{1}{2}t^2} & y(t) &= \pm y_0 \mathrm{e}^{-t} & z &= z_0
\end{aligned}
\tag{12}
$$

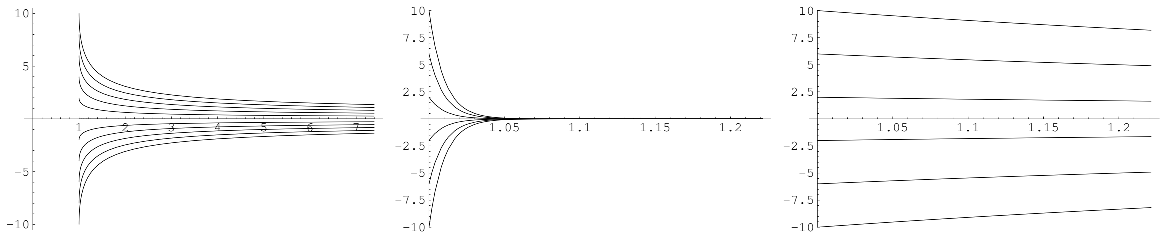

Figure 6: (left) Particle path (12) with parameters: $x_0 = 1$,$y_0 = -10, , -6, -2, 2, 6, 10$, $z_0 = 0$, streamlines (13) at (center) $t = 0.01$ (right) $t = 1$ with same initial values for x_0, y_0 and z_0.

Streamlines

$$\begin{aligned}
\frac{dx}{ds} &= xt & \frac{dy}{ds} &= -y & \frac{dz}{ds} &= 0 \\
\pm x(s) &= x_0 \mathrm{e}^{st} & \pm y(s) &= y_0 \mathrm{e}^{-s} & z(s) &= z_0 \\
\pm x(s) &= x_0 \mathrm{e}^{st} & s &= -\ln \frac{y}{y_0} & z &= z_0 \\
\pm x(y) &= x_0 \mathrm{e}^{-t \ln(y/y_0)} = \frac{x_0 y_0^t}{y^t} & & & & \qquad (13)
\end{aligned}$$