1. Verify that the example flow $\boldsymbol{v} = (ax, -ay, 0)$ satisfies the continuity equation with constant a and constant density. Determine the stream function ψ. Discuss the particle paths based on ψ.
Solution:
Continuity equation: When density is constant $\rho(t, \boldsymbol{r}) = \rho$, the continuity equation $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{v}) = 0$ reduces to $\nabla \cdot \boldsymbol{v} = 0$. In the case of the example flow $\boldsymbol{v} = (ax, -ay, 0)$, we get that

$$\nabla \cdot \boldsymbol{v} = \left(\boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial x} + \boldsymbol{k}\frac{\partial}{\partial x} \right) \cdot (ax\boldsymbol{i} + ay\boldsymbol{j}) = a - a = 0.$$

The stream function ψ is connected to the velocity components via the equations

$$ax = u = \frac{\partial \psi}{\partial y} \qquad\qquad -ay = v = -\frac{\partial \psi}{\partial y}.$$

The first gives that $\psi(x, y) = axy + f(x)$, which is plugged in to the second giving that $f'(x) = 0$. Thus, $\psi(x, y) = axy + C$, where C is a constant.
The stream lines: The stream function is constant along stream lines. Thus, by solving equation $\psi(x, y) = na + C$, we will find out the stream lines

$$y = \frac{n}{x},$$

which is a set of hyperbolas similarly as in the exercise 1.4(a) (see Fig. 5 of the solution set 1). The stream lines and the particle paths are the same thing now when the flow is time-independent, that is, steady flow.

2.
   (a) Using the method explained in the book or in the appendix of the lecture notes calculate $\nabla \cdot (f(r)\hat{\mathbf{r}})$ in a spherical system of coordinates.
   (b) Let $\mathbf{v} = mr^{-2}\hat{\mathbf{r}}$ in a spherical system of coordinates. Show that $\nabla \cdot \mathbf{v} = 0$ except at origin O. Let S be *any* smooth surface surrounding O. Show that volume flows through S at rate $4\pi m$. What is the corresponding result if O lies on S?

Solution:
(a) In the spherical coordinates, the nabla operator is

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\lambda}}\frac{1}{r \sin\theta}\frac{\partial}{\partial \lambda}$$

The unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\lambda}}$ are pointwise orthogonal but they depend on the spherical coordinates r, θ and λ, as we have seen in the exercise 1.3.

$$\begin{aligned}
\nabla \cdot (f(r)\hat{\mathbf{r}}) &= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{\lambda}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \lambda}\right) \cdot (f(r)\hat{\mathbf{r}}) \\
&= \hat{\mathbf{r}} \cdot \frac{\partial (f(r)\hat{\mathbf{r}})}{\partial r} + \frac{1}{r}\hat{\boldsymbol{\theta}} \cdot \frac{\partial (f(r)\hat{\mathbf{r}})}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial (f(r)\hat{\mathbf{r}})}{\partial \lambda} \\
&= f'(r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} + \frac{f(r)}{r}\hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} + \frac{f(r)}{r\sin\theta}\hat{\boldsymbol{\lambda}} \cdot \sin\theta\hat{\boldsymbol{\lambda}} \\
&= f'(r) + \frac{f(r)}{r} + \frac{f(r)\sin\theta}{r\sin\theta} = f'(r) + \frac{2f(r)}{r} = \frac{1}{r^2}\partial_r(r^2 f(r))
\end{aligned}$$

On the third line, the identity derived in the exercise 1.3, $\partial_\theta \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}}$ and another identity, which could have been derived in the same exercise, $\partial_\lambda \hat{\mathbf{r}} = \sin\theta\hat{\boldsymbol{\lambda}}$ are applied.

(b) Using the above lemma, it is easy to show that $\nabla \cdot \mathbf{v} = \mathbf{0}$ for the radially symmetric velocity field $\mathbf{v}$: $\nabla \cdot \mathbf{v} = -2mr^{-3} + 2mr^{-3} = 0$, except at the origin O, where the vector field $\mathbf{v}$ diverges. Let S be any smoooth surface surrounding origin O. The flow F through this surface reads

$$\begin{aligned}
F &= \int_S \mathbf{v} \cdot \mathrm{d}\mathbf{S} = \int_{\mathbf{V}} \nabla \cdot \mathbf{v}\mathrm{d}\mathbf{V} && \text{using Gauss' theorem} \\
&= \lim_{r\to 0} \int_{V_r} \nabla \cdot \mathbf{v}\mathrm{d}V + \underbrace{\int_{V\setminus V_r} \nabla \cdot \mathbf{v}\mathrm{d}V}_{=0} && V_r \text{ is r-radius ball around O} \\
&= \lim_{r\to 0} \int_{S_r} \mathbf{v} \cdot \mathrm{d}\mathbf{S} && \text{Volume } V \setminus V_r \text{ does not} \\
& && \text{contain the origin O: } \nabla \cdot \mathbf{v} = 0 \\
&= \lim_{r\to 0} \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\lambda \frac{m}{r^2}\sin\theta r^2 \\
&= \lim_{r\to 0} 4\pi m = 4\pi m
\end{aligned}$$

In the volume integration, one can divide the integrated volume in the two parts as one can do in one dimensional integration: $\int_V = \int_{V_1} + \int_{V_2}$ if $V = V_1 \cup V_2$. The Gauss' theorem can be applied only for smooth surfaces.
If the origin O lies on the surface S, the procedure goes as above, but one cannot draw full sphere around the O but a hemisphere. The result is then $F = 2\pi m$.

3.

(a) Calculate $D\mathbf{v}/Dt$ for the steady two-dimensional circular flow $\mathbf{v} = f(r)\hat{\boldsymbol{\theta}}$. Does your result fit in with particle dynamics?

(b) Water flows along a pipe whose area of cross-section $A(x)$ varies slowly with the coordinate x along the pipe. Express the mass flow at x using $A(x)$, the density

ρ and the velocity $\boldsymbol{v}_{\text{ave}}(x) \approx v_{\text{ave}}\boldsymbol{i}$, which is averaged over the cross section of the pipe. Use the conservation of mass to determine $\boldsymbol{v}_{\text{ave}}(x)$ in the pipe, and calculate the acceleration of a particle moving with this averaged velocity.

Solution:

(a) In the polar coordinates, we have the circular flow $\mathbf{v} = f(r)\hat{\boldsymbol{\theta}}$ and the nabla operator $\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}$. The convective derivative operator is

$$\frac{D(\cdot)}{Dt} = \frac{\partial(\cdot)}{\partial t} + (\mathbf{v}\cdot\nabla)(\cdot).$$

Let us calculate

$$\begin{aligned}
\frac{D\mathbf{v}}{Dt} &= \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \\
&= \frac{\partial(f(r)\hat{\boldsymbol{\theta}})}{\partial t} + \left(f(r)\hat{\boldsymbol{\theta}}\cdot\left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\right)\right) f(r)\hat{\boldsymbol{\theta}} \\
&= \left(\frac{f(r)}{r}\frac{\partial}{\partial\theta}\right) f(r)\hat{\boldsymbol{\theta}} = \frac{f^2(r)}{r}\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\frac{f^2(r)}{r}\hat{\mathbf{r}} = -\frac{|\mathbf{v}|^2}{r}\hat{\mathbf{r}}
\end{aligned}$$

The solution $\frac{D\mathbf{v}}{Dt} = -\frac{|\mathbf{v}|^2}{r}\hat{\mathbf{r}}$ fits very well with particle dynamics in a circular flow: $\frac{D\mathbf{v}}{Dt}$ is the acceleration that the particles experience and $-\frac{|\mathbf{v}|^2}{r}\hat{\mathbf{r}}$ is the central acceleration know from the mechanics.

(b) *The mass flow at x:* In general, the velocity may depend on coordinates (x, y, z) in the pipe, e.g. the velocity is probably highest in the center of the pipe and vanishes near the pipe walls. The local mass flow is $\boldsymbol{m}(x, y, z) = \rho(x, y, z)\boldsymbol{v}(x, y, z)$. We are now interested in the total mass flow $\boldsymbol{M}$, calculated as the integral of the local mass flow $\boldsymbol{m}$ over the the cross section A of the pipe:

$$\begin{aligned}
\boldsymbol{M}(x) &= \iint_A \rho(x, y)\boldsymbol{v}(x, y, z)\mathrm{d}y\,\mathrm{d}z \\
&= \rho\left(\iint_A \mathrm{d}y\,\mathrm{d}z\right)\frac{\int_A \boldsymbol{v}(x, y, z)\mathrm{d}y\,\mathrm{d}z}{\iint_A \mathrm{d}y\,\mathrm{d}z} \\
&= \rho A(x)\boldsymbol{v}_{\text{ave}} = \rho A(x) v_{\text{ave}}\boldsymbol{i}
\end{aligned}$$

The pipe is assumed to be aligned along the x-axis and that the fluid is incompressible, i.e. constant $\rho(x, y, z) = \rho$.

The averaged velocity: Consider a section of the pipe between x_0 and x, with volume V surrounded by the surface S, which consists of the walls of the pipe and of the cross-sectional surfaces $A_0 = A(x_0)$ and $A = A(x)$. Having incompressible fluid, no flow trough the walls of the pipe and the averaged mass flow in the x-direction, we express the conservation of the mass as

$$-\boldsymbol{M}(x_0) + \boldsymbol{M}(x) = 0.$$

The flow into the volume must come out. Then, the expression for the averaged velocity reads

$$\boldsymbol{v}_{\text{ave}}(x) = \frac{A_0 v_0}{A(x)}\boldsymbol{i}.$$

In the other words, the mass flow $\boldsymbol{M}$ is constant in the pipe, $\boldsymbol{M}(x) = c\boldsymbol{i}$.
Acceleration of a particle moving with the averaged velocity $\mathbf{v}(x)$:

$$\begin{aligned}\mathbf{a}(x) &= \frac{D\mathbf{v}_{\text{ave}}}{Dt} = \frac{\partial \mathbf{v}_{\text{ave}}}{\partial t} + (\mathbf{v}_{\text{ave}} \cdot \nabla)\mathbf{v}_{\text{ave}} \\ &= \left(\frac{A_0 v_0}{A(x)}\frac{\partial}{\partial x}\right)\frac{A_0 v_0}{A(x)}\boldsymbol{i} = -\frac{A_0^2 v_0^2 A'(x)}{A^3(x)}\boldsymbol{i} = -v_{\text{ave}}^2\frac{A'(x)}{A(x)}\boldsymbol{i}\end{aligned}$$

A note about the continuity equation and the conservation of the mass:
In the lectures, several equivalent forms of the continuity equation were given:

Global formulation	$\dfrac{\mathrm{d}M_v(t)}{\mathrm{d}t} = -\displaystyle\int_S \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$
Local formulation	$\dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho\boldsymbol{v}) = 0.$

Global formulation describes the conservation of the mass in global sense: the change of the mass $\mathrm{d}M_v(t)/\mathrm{d}t$ in some volume V must be equal to the flow $-\int_S \rho\boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}$ through surface S into V. On the other hand, the local formulation considers the mass flow $\rho\boldsymbol{v}$ and change of the density *at a specific point* x: the derivatives with respect to time ∂_t and spatial coordinates ∇ are calculated in the neighborhood of the point x, and the equation must hold at all points. In the solution above, we used the global formulation of the mass conservation in one dimension.

4. A flow around a cylinder can be described by the stream function

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta,$$

where U is a constant and a denotes the radius of the cylinder.
(a) Show that there is no flow through the surface $r = a$ of the cylinder.
(b) Calculate the tangential velocity v_θ on the surface of the cylinder.
(c) Find the stream lines corresponding to $\psi = naU$ (n integer) by calculating their positions when $x \to \infty$ and at $x = 0$, and sketching the rest.

Solution:
(a) The situation should be considered in principle in three dimensional cylindrical polar coordinates, but there is no dependence on the z component in the stream function or in the flow, thus the sitation is reduced to the two dimensional polar coordinates. As shown in lectures the flow $\boldsymbol{v}$ is written using stream function $\psi = U(r - \frac{a^2}{r})\sin\theta$:

$$\begin{aligned}\boldsymbol{v}(r,\theta) &= \hat{\mathbf{r}}\frac{1}{r}\frac{\partial \psi}{\partial \theta} - \hat{\boldsymbol{\theta}}\frac{\partial \psi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta\hat{\mathbf{r}} - U\left(1 + \frac{a^2}{r^2}\right)\sin\theta\hat{\boldsymbol{\theta}} \\ &= v_r(r,\theta)\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}\end{aligned}$$

The flow through the surface of the cylinder is reduced to the flow through a circle of radius a. The radial part of the flow vanishes at the surface of the circle: $v_r(a,\theta) = 0$, thus there is no flow through the cylinder

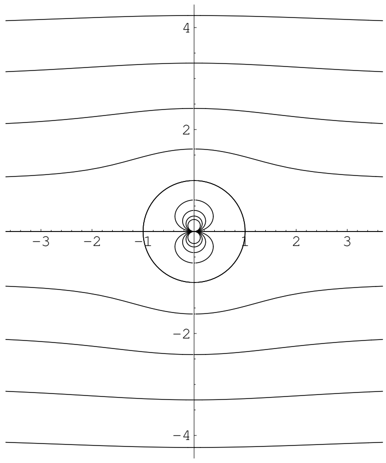

Figure 1: The streamlines around and, for the curiosity, inside the radius 1 cylinder.

$$F = \int_{S_a} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S} = \int_0^{2\pi} \boldsymbol{v} \cdot (a\mathrm{d}\theta\hat{\mathbf{r}}) = \int_0^{2\pi} v_r(a,\theta)a\mathrm{d}\theta = 0 \tag{1}$$

(b) Above, we notice that $v_\theta(r,\theta) = -U(1+\frac{a^2}{r^2})\sin\theta$. Thus, on the surface of the cylinder $r = a$ and

$$v_\theta(r = a, \theta) = -2U\sin\theta.$$

The message is that tangential velocity does not vanish on the surface of the cylinder and this flow cannot be realistic near the cylinder.

(c) We will now use the fact that *the stream function ψ is constant along a stream line.* We can choose this constant rather arbitrarily, and in this exercise we have taken it to be naU, where n is integer. We solve the trajectories along which ψ is constant. *Stream lines at $x = 0$* correspond to $\theta = \pi/2$ and $r = \sqrt{x^2+y^2} = y$ using these:

$$anU = \psi \quad \Rightarrow \quad anU = U\left(r - \frac{a^2}{r}\right)\sin\theta \quad \Rightarrow \quad an = y - \frac{a^2}{y} \quad \Rightarrow$$

$$y^2 - any - a^2 = 0 \quad \Rightarrow \quad y = a\left(\frac{n}{2} \pm \sqrt{\left(\frac{n}{2}\right)^2 + 1}\right) \qquad n = 0, 1, 2, 3, \ldots$$

We should also considered the possibility that $\theta = -\pi/2$ and $\sin\theta = -1$ but it gives just symmetric result for $y < 0$.

Stream lines at $x = \infty$ correspond to $\sin\theta = \frac{y}{r}$ and $r \to \infty$.Using these:

$$anU = \psi \quad \Rightarrow \quad anU = U\left(r - \frac{a^2}{r}\right)\sin\theta \quad \Rightarrow \quad an = \left(r - \frac{a^2}{r}\right)\frac{y}{r} \quad \Rightarrow$$

$$an = y - \frac{a^2}{r^2} \quad \Rightarrow \quad y = na \quad n = 0, 1, 2, 3, \ldots$$