1. For applications later in this course, go through the solution of the Laplace equation given in the appendix C of the lecture notes. Find $\phi(x, y)$ for the cases
   a) $f(y) = C\delta(y - \frac{a}{2})$,
   b) $f(y) = C \sin \frac{\pi y}{a}$.

   Hint: Function $\delta(y - \frac{a}{2})$ denotes Dirac delta (δ) function at $y = \frac{a}{2}$. Generally, δ function is defined with help of integration:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\mathrm{d}x = f(x_0).$$

Solution:
Note that in the appendix the following result has been used:

$$\int_0^a \sin(k_n y)\sin(k_m y)\, dy = \frac{a}{2}\delta_{nm},$$

where $k_n = n\pi/a$, which is obtained by change of variables from the more familiar orthogonality theorem

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\, dx = \pi\delta_{nm}.$$

Also note that the values of n are limited to $n = 1, 2 \ldots$, and $n = 0, -1, -2 \ldots$ are NOT accepted, since the exponential $e^{-k_n x}$ must vanish when $x \to \infty$.
Now, $\phi(x, y)$ can be presented as

$$\phi(x, y) = \sum_{n=1}^{\infty} D_n e^{-k_n x} \sin k_n y,$$

where

$$D_n = \frac{2}{a}\int_0^a \sin(k_n y) f(y)\, dy.$$

a) Now $f(y) = C\delta(y - \frac{a}{2})$, where the delta function δ is such that

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0)$$

for any (well behaving) function $f(x)$. Using this we get

$$D_n = \frac{2}{a}C\int_0^a \sin(k_n y)\delta\left(y - \frac{a}{2}\right)\, dy = \frac{2}{a}C\sin\left(n\frac{\pi}{2}\right).$$

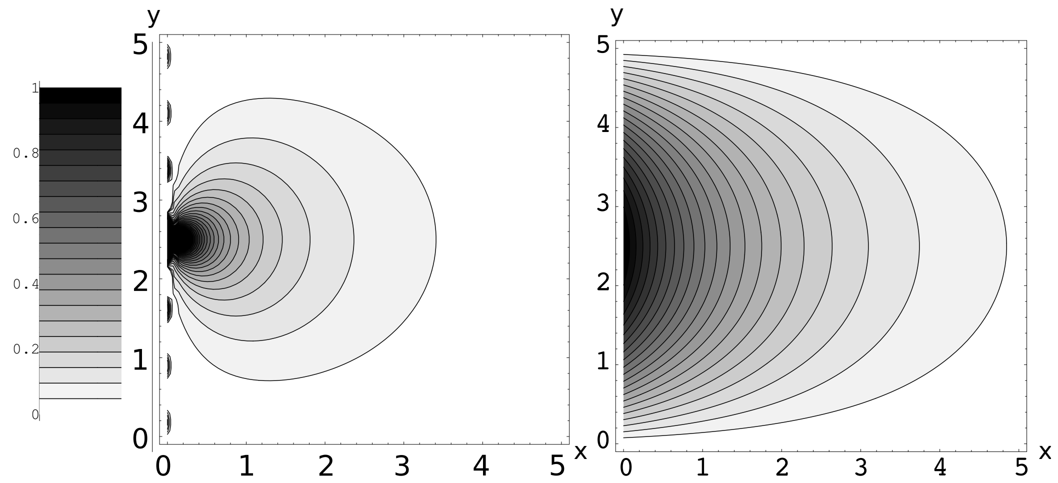

Figure 1: Contourplot of the solutions (1) and (2) at center and right panels, respectively. The left panel is the scale, the distance of contourlines is 0.05. Parameters are $C = 1$ and $a = 5$. In the center panel, there is seen oscillation near y-axis, it is unphysical and originates from the truncation of the infinite $\sum_{n=0}^{\infty}$ up to first 400 terms. Plotting of infinite sum is hard.

For even n, $\sin(n\frac{\pi}{2}) = 0$, while for odd n, we get $\sin(n\frac{\pi}{2}) = (-1)^{(n-1)/2}$. Thus, the solution [visualized in Fig. 1(center)] is

$$\phi(x, y) = \frac{2}{a} C \sum_{n=1}^{\infty} (-1)^{n-1} e^{-k_{2n-1}x} \sin(k_{2n-1}y). \tag{1}$$

b) Now $f(y) = C \sin \frac{\pi y}{a}$, so

$$D_n = \frac{2}{a} C \int_0^a \sin(k_n y) \sin(k_1 y)\, dy = C\delta_{n1},$$

i.e. $D_1 = C$ and $D_n = 0$ for $n \neq 1$. We then end up with

$$\phi(x, y) = Ce^{-\pi x/a} \sin(\pi \frac{y}{a}), \tag{2}$$

which is also visualized in Fig. 1(right).

2. Given the flow

$$\mathbf{v} = (3z + 4x, -5y, -2x + z),$$

calculate the vorticity and the symmetric and antisymmetric parts of $\partial v_i/\partial x_j$.
Solution:
We denote $\mathbf{v} = (v_1, v_2, v_3)$ (note that $\nabla \cdot \mathbf{v} = 0$). The vorticity is

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & v_3 \end{vmatrix} = (2 + 3)\mathbf{j} = 5\mathbf{j}.$$

The symmetric (e) and antisymmetric (r) tensors are calculated from the equations

$$e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

$$r_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right).$$

First we calculate the tensor

$$\frac{\partial v_i}{\partial x_j} = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{pmatrix} = \begin{pmatrix} 4 & 0 & 3 \\ 0 & -5 & 0 \\ -2 & 0 & 1 \end{pmatrix}.$$

It is then easy to calculate

$$e_{ij} = \begin{pmatrix} 4 & 0 & \frac{1}{2}(3-2) \\ 0 & -5 & 0 \\ \frac{1}{2}(-2+3) & 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & \frac{1}{2} \\ 0 & -5 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix},$$

and

$$r_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2}(3+2) \\ 0 & 0 & 0 \\ \frac{1}{2}(-2-3) & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{5}{2} \\ 0 & 0 & 0 \\ -\frac{5}{2} & 0 & 0 \end{pmatrix}.$$

3. Poiseuille flow in a pipe has velocity components

$$u = v = 0, \; w = b(a^2 - x^2 - y^2),$$

where $\mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

a) Calculate $\nabla \cdot \mathbf{v}$ and $\nabla \times \mathbf{v}$.

b) Calculate the symmetric and antisymmetric parts of $\partial v_i/\partial x_j$.

c) Find the eigenvalues and (eigenvectors) principal axes of the symmetric part.

d) Express the vorticity in the cylindrical polar coordinates and discuss the direction of the vorticity in terms of the slipping of layers of fluid over each other.

Solution:

a) Divergence of the velocity field $\boldsymbol{v} = u\boldsymbol{i} + \boldsymbol{v}j + w\boldsymbol{j}$

$$\nabla \cdot \mathbf{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Curl of the velocity field in the Cartesian coordinates

$$\begin{aligned} \nabla \times \mathbf{v} &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k} \\ &= \frac{\partial w}{\partial y}\boldsymbol{i} - \frac{\partial w}{\partial x}\boldsymbol{j} = -2by\boldsymbol{i} + 2bx\boldsymbol{j} \end{aligned}$$

and in the cylindrical polar coordinates where $y = r\sin\theta$ and $x = r\cos\theta$

$$\nabla \times \mathbf{v} = -2br\sin\theta\boldsymbol{i} + 2br\cos\theta\boldsymbol{j} = 2br(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) = 2br\hat{\boldsymbol{\theta}}.$$

b)

$$\nabla\boldsymbol{v} = \frac{\partial v_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2bx & -2by & 0 \end{pmatrix}$$

Symmetric part is defined as $e_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$ and antisymmetric as $r_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$ thus they are

$$\mathsf{e} = \begin{pmatrix} 0 & 0 & -bx \\ 0 & 0 & -by \\ -bx & -by & 0 \end{pmatrix} \qquad \mathsf{r} = \begin{pmatrix} 0 & 0 & bx \\ 0 & 0 & by \\ -bx & -by & 0 \end{pmatrix}.$$

c) Eigenvalues λ and eigenvectors $\boldsymbol{a}$ solve the vector-matrix equation $\mathsf{e}\boldsymbol{a} = \lambda\boldsymbol{a}$ which has alternative formulation $(\mathsf{e} - I\lambda)\boldsymbol{a} = 0$. This eigenvalue problem has nontrivial solution ($\boldsymbol{a} \neq 0$) iff $\det(\mathsf{e} - I\lambda) = 0$ thus one have to solve equation

$$\begin{vmatrix} -\lambda & 0 & -bx \\ 0 & -\lambda & -by \\ -bx & -by & -\lambda \end{vmatrix} = 0 \tag{3}$$

which reduces to the form $-\lambda^3 + \lambda b^2(x^2 + y^2) = 0$ having solutions

$$\lambda_0 = 0, \qquad \lambda_1 = b\sqrt{x^2 + y^2}, \qquad \lambda_2 = -b\sqrt{x^2 + y^2}.$$

These are the eigenvalues of the matrix e. Eigenvectors $\boldsymbol{a}^{(i)}$ are solved from the equation $(\mathsf{e} - \mathsf{I}\lambda_i)\boldsymbol{a}^{(i)} = 0$. It is also convenient to normalize eigenvectors such that $|\boldsymbol{a}^{(i)}| = 1$. For the first, $\lambda_0 = 0$ then

$$\begin{pmatrix} 0 & 0 & -bx \\ 0 & 0 & -by \\ -bx & -by & 0 \end{pmatrix} \begin{pmatrix} a_x^{(0)} \\ a_y^{(0)} \\ a_z^{(0)} \end{pmatrix} = 0$$

which has normalized solution

$$\boldsymbol{a}^{(0)} = \begin{pmatrix} -\frac{y}{\sqrt{x^2+y^2}} \\ +\frac{x}{\sqrt{x^2+y^2}} \\ 0 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ +\cos\theta \\ 0 \end{pmatrix} = \sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} = \hat{\boldsymbol{\theta}}$$

and then $\lambda_{1,2} = \pm b\sqrt{x^2 + y^2} = \pm br$ then we have

$$\begin{pmatrix} \mp b\sqrt{x^2 + y^2} & 0 & -bx \\ 0 & \mp b\sqrt{x^2 + y^2} & -by \\ -bx & -by & \mp b\sqrt{x^2 + y^2} \end{pmatrix} \begin{pmatrix} a_x^{(1,2)} \\ a_y^{(1,2)} \\ a_z^{(1,2)} \end{pmatrix} = 0$$

which has normalized solutions

$$\boldsymbol{a}^{(1,2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp\frac{x}{\sqrt{x^2+y^2}} \\ \mp\frac{y}{\sqrt{x^2+y^2}} \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \mp\cos\theta \\ \mp\sin\theta \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(\mp\hat{\mathbf{r}} + \boldsymbol{k}).$$

These eigenvectors $\boldsymbol{a}^{(1,2,3)}$ are the wanted principal axes. The meaning of the principal axes: As $\lambda_1 = b\sqrt{x^2+y^2} = br > 0$ a volume element stretches in the direction of the principal axis $\boldsymbol{a}_1$ and as $\lambda_2 = -br < 0$ the volume element squeezes in the direction of $\boldsymbol{a}_2$. In the direction of vector $\boldsymbol{a}_0$, it happens no transformation, since $\lambda_0 = 0$. Note that direction of the eigenvectors is not fixed: an eigenvector $-\boldsymbol{a} = \boldsymbol{b}$ satisfies the equation $\mathsf{e}\boldsymbol{b} = \lambda\boldsymbol{b}$ as well as eigenvector $\boldsymbol{a} = \boldsymbol{b}$.

d) The vorticity $\nabla \times \boldsymbol{v} = 2br\hat{\boldsymbol{\theta}}$ as is shown in (a).

Velocity profile in the $x - z$ plane is upsidedown parabola, shown in the Fig. 2. The minimum of the velocity is at the edge of the pipe $r = a$ and the maximum at the center $r = 0$. If approaching the center from the edge, the velocity increases as the slope $2br$ of the parabola.

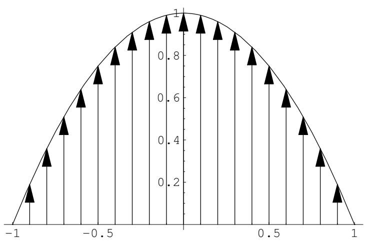

Figure 2: Velocity profile at $x - z$ plane.

Slipping of the flow layers If the pipe were split in circular shaped layers with thickness dr and radius r, then the velocity in $\boldsymbol{z}$ dirction would increase when approaching the center. Neighbour layers have different velocity and they are slipping with respect to each other. This is a similar to the case of shear flow discussed in the lectures with the help of the rotation of the small cross, but now in cylindrical coordinates. Similarly we can deduce that there is local rotation with angular velocity $\boldsymbol{w} = br\hat{\boldsymbol{\theta}}$.

Differential operator nabla ∇ is at elementary level defined at a point $\boldsymbol{r}$ and it's small neighbourhood. In the physical language, ∇ operations give only local information of the field they operate, $\nabla \times \boldsymbol{v}$ is the local vorticity of the flow. Global rotation must be deduced from the local vorticity, e.g., by integration.

Gedanken experiment Consider that you are going to put a small paddle wheel in the flow and fix only the point where the wheel stands. If the wheel starts to rotate then the value of vorticity $\nabla \times \boldsymbol{v}$ is non-zero. If the axis of the wheel is freely moving, it will set to same direction than the vector $\nabla \times \boldsymbol{v}$. By studying the behaviour of the paddle wheel in the whole flow, one can deduce the vorticity field, which in our case is $2br\hat{\boldsymbol{\theta}}$.

4. A vortex has the stream function $\psi = -C \ln \frac{r}{a}$. Calculate the vorticity outside of the line ($r = 0$) to show that $\nabla \times \boldsymbol{v} = 0$. Show, by using the Stokes' theorem, that the circulation $\kappa = \oint \mathbf{v} \cdot d\boldsymbol{l}$ for vortex flow is the same for any simple curve once around the origin (in the positive direction).
Solution: *Vorticity*: First, we calculate $\mathbf{v}$ from

$$\mathbf{v} = (\nabla\psi) \times \hat{\mathbf{k}} = \left(-\frac{C}{r}\hat{\mathbf{r}}\right) \times \hat{\mathbf{k}} = \frac{C}{r}\hat{\boldsymbol{\theta}}.$$

Then, we calculate $\nabla \times \boldsymbol{v}$ in the cylindrical coordinates outside of the center line $r = 0$:

$$\begin{aligned}\nabla \times \boldsymbol{v} &= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta} + \boldsymbol{k}\frac{\partial}{\partial z}\right) \times \left(\frac{C}{r}\hat{\boldsymbol{\theta}}\right) \\ &= \hat{\mathbf{r}} \times \left(-\frac{C}{r^2}\hat{\boldsymbol{\theta}}\right) + \hat{\boldsymbol{\theta}} \times \left(\frac{C}{r^2}\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta}\right) \\ &= \hat{\mathbf{r}} \times \left(-\frac{C}{r^2}\hat{\boldsymbol{\theta}}\right) + \hat{\boldsymbol{\theta}} \times \left(-\frac{C}{r^2}\hat{\mathbf{r}}\right) = 0.\end{aligned}$$

The local vorticity outside the center line vanishes. This was elegantly demonstrated in the video shown in the lectures by studying a small cross in a circular flow. *Circulation:* Now we can calculate the circulation for a circular path ℓ_1 around origin. For this we have $d\boldsymbol{l} = r\hat{\boldsymbol{\theta}}d\theta$, and therefore

$$\kappa = \oint_{\ell_1} \mathbf{v} \cdot d\boldsymbol{l} = \int_0^{2\pi} v_\theta r d\theta = 2\pi C.$$

We will then show that the results is the same for an arbitrary path ℓ_2 once around the origin. We will use the Stokes' theorem, which states

$$\iint_A \nabla \times \mathbf{v} \cdot d\mathbf{A} = \oint_l \mathbf{v} \cdot d\boldsymbol{l}.$$

We will construct a path ℓ that consists of the given path ℓ_2 in the positive direction, a circular path ℓ_1 around the origin inside ℓ_2, and two straight lines ℓ_3 and ℓ_4 in opposite directions, which connect ℓ_1 and ℓ_2, see Fig. 3. In other words

$$\oint_\ell = \oint_{\ell_2} + \int_{\ell_3} - \oint_{\ell_1} + \int_{\ell_4} = \oint_{\ell_2} - \oint_{\ell_1},$$

where the negative sign of $\oint_{\ell_1}$ comes from the fact that it is now directed in the negative direction, and the integrations over ℓ_3 and ℓ_4 cancel each other. Now the

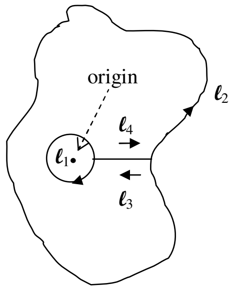

Figure 3: The path ℓ used in the integration.

area A inside the combined path ℓ does not contain the origin, so $\nabla \times \mathbf{v} = 0$ inside A, and from Stokes' theorem we get

$$\oint_{\ell} \mathbf{v} \cdot d\boldsymbol{l} = \oint_{\ell_2} \mathbf{v} \cdot d\boldsymbol{l} - \oint_{\ell_1} \mathbf{v} \cdot d\boldsymbol{l} = \iint_A \nabla \times \mathbf{v} \cdot d\mathbf{A} = 0,$$

or

$$\oint_{\ell_2} \mathbf{v} \cdot d\boldsymbol{l} = \oint_{\ell_1} \mathbf{v} \cdot d\boldsymbol{l} = 2\pi C,$$

which is what we wanted to show. Alternatively, one could use the method similar to that used in the previous exercise set, in connection of the Gauss' theorem (or vice versa).