1. A frame of reference which is accelerating (with respect to inertial frames) is used to describe an experiment. The acceleration has the constant value f in the $\mathbf{i}$ direction.
   (a) Show that an 'inertial force' $-\rho f\mathbf{i}$ acts on a fluid (per volume) and that a potential fx may be used.
   (b) Hence find the equilibrium water surface in an accelerated tank of water by taking into account also the gravity.

   Hint: It will be shown in the next lecture that the liquid surface corresponds to potential Φ=constant.

   Solution:

   (a) Let O be an inertial frame and O' the accelerating frame. Let $\mathbf{s}$ be the vector connecting the origins of the two frames at some instant. Then position vectors $\mathbf{r}$ and $\mathbf{r}'$ are related by $\mathbf{r} = \mathbf{s} + \mathbf{r}'$.
   Newtons second law states that $\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}$. Thus

$$\mathbf{F} = m\frac{d^2\mathbf{s}}{dt^2} + m\frac{d^2\mathbf{r}'}{dt^2} = \mathbf{F}_i + \mathbf{F}',$$

   or $\mathbf{F}' = \mathbf{F} - \mathbf{F}_i$. But now $\frac{d^2\mathbf{s}}{dt^2}$ is nothing but the given acceleration $\mathbf{a} = f\hat{\mathbf{i}}$, so the "inertial force" per unit volume is

$$-\frac{\mathbf{F}_i}{V} = -\frac{m}{v}f\hat{\mathbf{i}} = -\rho f\hat{\mathbf{i}}.$$

   Since $-\nabla(fx) = -f\hat{\mathbf{i}}$, the corresponding potential is fx.

   (b) The potential of the gravity is $\Phi_g = gz$ and the potential of the inertial force $\Phi_i = fx$. Using them, the water surface is given by

$$\Phi = \Phi_g + \Phi_i = gz + fx = \text{constant} = C \Rightarrow z = \frac{C}{g} - \frac{f}{g}x.$$

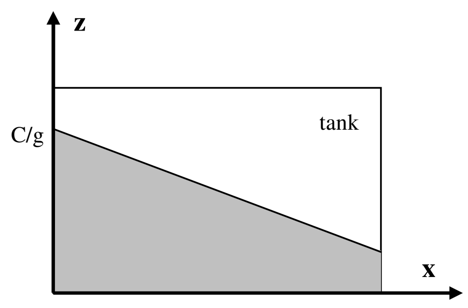

Figure 1: The surface of water in the tank.

2. Show that the relation $\delta A_1 = \mathbf{n} \cdot \mathbf{i} \delta A$ used in the lectures is valid.

   Hint: Express $\mathbf{n}\delta A$ as a cross product of two vectors.
   Solution:

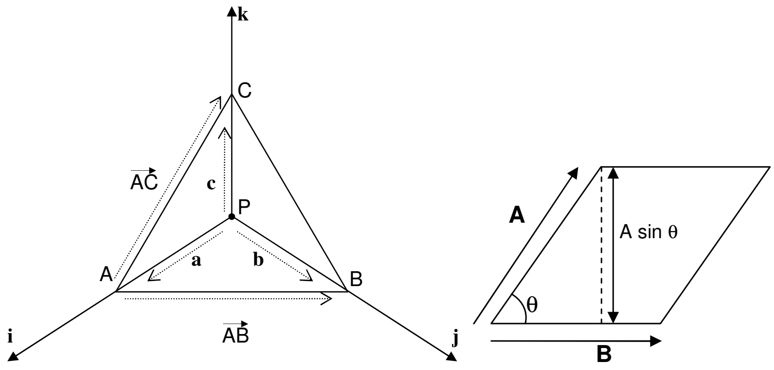

Figure 2: LEFT: Definition of vectors **a**, **b**, **c**, AB, and AC. RIGHT: The area of a parallelogram spanned by vectors **A** and **B** is $|\mathbf{A} \times \mathbf{B}| = AB \sin\theta$, where θ is the angle between the vectors. The area of a triangle spanned by **A** and **B** is half of this.

We define $\mathbf{a} = a\mathbf{i}$, $\mathbf{b} = b\mathbf{j}$, and $\mathbf{c} = c\mathbf{k}$, where a is the distance between points A and P, i.e. $a = AP$. Correspondingly $b = BP$, and $c = CP$. Now

$$\overrightarrow{AC} = -\mathbf{a} + \mathbf{c}, \quad \text{and} \quad \overrightarrow{AB} = -\mathbf{a} + \mathbf{b}.$$

From the "right-handed three-finger rule", we find for the surface normal of δA

$$\hat{\mathbf{n}} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{|\overrightarrow{AB} \times \overrightarrow{AC}|}.$$

The area δA can be expressed as $\delta A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$. Thus we have

$$\delta A \hat{\mathbf{n}} = \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC} = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = \frac{1}{2}(bc\mathbf{i} + ac\mathbf{j} + ab\mathbf{k}).$$

Now for the area δA_1

$$\delta A_1 = \frac{1}{2}|\mathbf{b} \times \mathbf{c}| = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = \frac{1}{2}bc = \delta A \hat{\mathbf{n}} \cdot \mathbf{i}.$$

3. Show that the normal component of the surface force vector is

$$\sigma_{ij} n_j n_i$$

per area, and find an expression for the tangential force on area dS (i.e. the force parallel to the surface).

The solution is written simoultanous by using vector $\boldsymbol{A}$ and vector component A_i notation.

A tensor $\bar{\bar{\Sigma}}$ is thought to be constructed from two vectors: $\bar{\bar{\Sigma}}= \boldsymbol{ab}$, without dot product or anything. Now one can take dot products with the tensor from right and left:

$$\bar{\bar{\Sigma}} \cdot \boldsymbol{c} = \boldsymbol{ab} \cdot \boldsymbol{c} \qquad\qquad \boldsymbol{c} \cdot \bar{\bar{\Sigma}} = \boldsymbol{c} \cdot \boldsymbol{ab}.$$

The tensor $\bar{\bar{\Sigma}}$ has ij component of form: $\Sigma_{ij} = a_i b_j$ and the above dot products reads as

$$\Sigma_{ij} c_j = a_i b_j c_j \qquad\qquad c_i \Sigma_{ij} = c_i a_i b_j.$$

You can see this construction also in $\nabla \boldsymbol{v}$ which is in component stands $\frac{\partial v_i}{\partial x_j}$.

The surface element vector $\mathrm{d}\boldsymbol{S}$ has expression

$$\text{in vector-matrix notation:} \quad \mathrm{d}\boldsymbol{S} = \hat{\mathbf{n}} \mathrm{d}S$$
$$\text{in coordinate notation:} \quad \mathrm{d}S_j = n_j \mathrm{d}S$$

The surface force vector $\mathrm{d}\boldsymbol{F}$ reads as

$$\text{in vector-matrix notation} \quad \mathrm{d}\boldsymbol{F} = \bar{\bar{\sigma}} \cdot \mathrm{d}\boldsymbol{S} = \bar{\bar{\sigma}} \cdot \boldsymbol{n} \mathrm{d}S$$
$$\text{in coordinate notation} \quad \mathrm{d}F_i = \sigma_{ij} \mathrm{d}S_j = \sigma_{ij} n_j \mathrm{d}S$$

where $\bar{\bar{\sigma}}$ is the stress tensor and $\hat{\mathbf{n}}$ the normal vector of the surface. The projection of a vector $\boldsymbol{A}$ onto unit vector $\hat{\mathbf{n}}$ is given by dot product: $A^{(n)} = \hat{\mathbf{n}} \cdot \boldsymbol{A} = n_i A_i$. Thus the normal component $f^{(n)}$ of the surface force vector is

$$\text{in vector-matrix notation} \quad f^{(n)} = \hat{\mathbf{n}} \cdot \frac{\mathrm{d}\boldsymbol{F}}{\mathrm{d}S} = \hat{\mathbf{n}} \cdot \bar{\bar{\sigma}} \cdot \hat{\mathbf{n}}$$
$$\text{in coordinate notation} \quad f^{(n)} = n_i \frac{\mathrm{d}F_i}{\mathrm{d}S} = n_i \sigma_{ij} n_j = \sigma_{ij} n_i n_j.$$

The projection $f^{(n)}$ is just a bare number. Now the normal component of the surface force vector reads $\boldsymbol{f}^{(N)} = f^{(n)} \hat{\mathbf{n}}$ or $f_i^{(N)} = f^{(n)} n_i$ and the remaining part of the force is the tangential part

$$\text{vector-matrix notation} \quad \boldsymbol{f}^{(T)} = \boldsymbol{f} - \boldsymbol{f}^{(N)} = (\bar{\bar{\sigma}} - f^{(n)} \bar{\bar{1}}) \cdot \hat{\mathbf{n}}$$
$$\text{coordinate notation} \quad f_i^{(T)} = f_i - f_i^{(N)} = \sigma_{ij} n_j - f^{(n)} n_i = \sigma_{ij} n_j - (n_k \sigma_{kl} n_l) n_i.$$

4. Let us assume that the Earth's gravitational field is isotropic.

   a) Show that due to Earth's rotation the ocean surface varies as

$$\delta r = \frac{\omega^2 R^2 \sin^2\theta}{2g}, \tag{1}$$

   where R is the radius and ω is the angular velocity of Earth and θ is the polar angle.

   b) How could you justify the use of constant radius R in the right hand side of equation (1)?

   c) Compare the results with measured values

$$\begin{aligned} R_{pole} &= 6\,356\,912\,\mathrm{m}, \\ R_{equator} &= 6\,378\,388\,\mathrm{m}. \end{aligned}$$

   What could cause the difference?

Hint: Liquid surface corresponds to Φ=constant.

Solution:

a) Let's consider a point $\boldsymbol{r}$ near the surface of Earth at the polar angle θ, and at distance $r = R + \delta r$ from Earth's center, see Fig. 3. Here, R is the (pole) radius of Earth, and δr is a small deviation.
The coordinate system at that point is such that r is perpendicular to the surface of Earth. Let's assume that the length scales considered are small respect the radius R of Earth. Then the gravitational force is $\boldsymbol{F}_g = -gr\hat{\mathbf{r}}$ and corresponding potential $\Phi_g = gr$, ($\boldsymbol{F}_g = -\nabla\Phi$). If this were not the case, we would use $1/r$ gravitational potential.

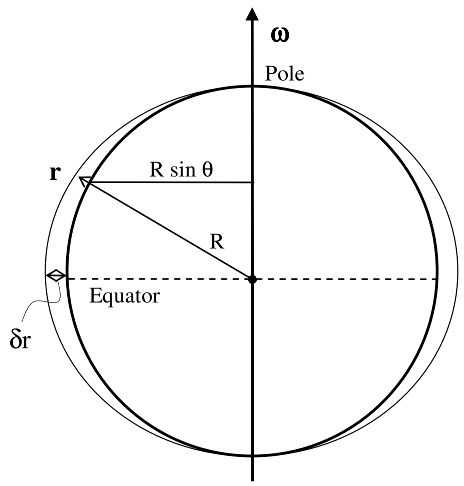

Figure 3: Schematics of the notations

For the rotational potential, we notice that the axis of rotation is at distance $r\sin\theta = (R+\delta r)\sin\theta$ from the point considered (see Fig. 3). The rotation is taken into account by centrifugal force: $\boldsymbol{F}_r = \omega^2 r\boldsymbol{r}$ and $\Phi_r = -\frac{1}{2}\omega^2 r^2$ ($\boldsymbol{F}_r = -\nabla\Phi$). The total potential is $\Phi = gr - \frac{1}{2}\omega^2 r^2 \sin^2\theta$. The surface of liquid follows a constant potential $\Phi = C$. We find the appropriate constant by considering the pole, $\theta = 0$, and setting there $r = R_{\text{pole}}$ ($\delta r = 0$), i.e. $C = gR_{\text{pole}}$. Inserting this to the expression for the total potential gives

$$\delta r = r - R_{\text{pole}} = \frac{\omega^2 r^2 \sin^2\theta}{2g} \approx \frac{\omega^2 R^2 \sin^2\theta}{2g}. \tag{2}$$

b) In the last form, we have approximated $r \approx R + \delta r$. Here R can be taken to be R_{pole}, although it makes little difference if we use R_{equator} instead. Using $R \approx 6 \cdot 10^6$ m in (2) we get $\delta r = 10900\,\text{m} \approx 10^4$ m at the equator, so

$$\begin{aligned}(R+\delta r)^2 = R^2 + 2R\delta r + (\delta r)^2 = R^2(1 + 2\frac{\delta r}{R} + \left(\frac{\delta r}{R}\right)^2)\\ \approx R^2(1 + 3\cdot 10^{-3} + 3\cdot 10^{-4}) \approx R^2,\end{aligned}$$

and corresponding change in δr is few tens of meters.

c) From equation (2), we get at equator $\theta = \pi/2$ that $\delta r = r - R_{\text{pole}} = \frac{\omega^2 R^2}{2g} = 10\,892$ m but from experimental values $R_{\text{eq}} - R_{\text{pole}} = 21\,476$ m. Sources of error may be
   (i) $g = MG/r^2$ is not constant, since the surface we consider is not a perfect sphere. We should use $\Phi_G = -\frac{MG}{r}$. This only causes a little difference.
   (ii) Since the surface of Earth is not idealized sphere, the gravitational field is not isotropic : $\boldsymbol{F}_g \not\parallel \hat{\mathbf{r}}$.

See more, for example, A. L. Fetter, and J. D. Walecka, *Theoretical Mechanics of Particles and Continua*, 2003, page 48.