1. Derive the formulae: a) $\int_S f d\mathbf{S} = \int_V \nabla f dV$, and b) $\int_S A_{ij} dS_j = \int_V \frac{\partial A_{ij}}{\partial x_j} dV$. [Hint: Multiply by a constant vector and use the divergence theorem.] **Solution**: The (Gauss') divergence theorem states:

$$\int_s \mathbf{A} \cdot d\mathbf{S} = \int_V \nabla \cdot \mathbf{A}\, dV, \tag{1}$$

or in the component form

$$\int_S A_i dS_i = \int_V \frac{\partial A_i}{\partial x_i}\, dV. \tag{2}$$

a) (Version 1) If we choose $A_i = f \;\; \forall i$, i.e. $\mathbf{A} = f\boldsymbol{i} + f\boldsymbol{j} + f\boldsymbol{k}$, then, using eq. (2), we get seperately for each component i that

$$\int_S f dS_i = \int_V \frac{\partial f}{\partial x_i}\, dV$$

which can be further written in the vector form as $\int_S f d\mathbf{S} = \int_V \nabla f dV$,.
(Version 2) Let's multiply f with a constant vector $\boldsymbol{e}$, for this vector

$$\int_S (f\boldsymbol{e}) \cdot \mathrm{d}\boldsymbol{S} = \int_V \nabla \cdot (f\boldsymbol{e})\, dV = \int_V \boldsymbol{e} \cdot (\nabla f)\, dV$$

Now, we choose first $\boldsymbol{e} = \boldsymbol{i}$, then $\boldsymbol{e} = \boldsymbol{j}$ and for the last $\boldsymbol{e} = \boldsymbol{k}$. This procedure gives the end result $\int_S f dS_i = \int_V \frac{\partial f}{\partial x_i}\, dV$.

b) (Version 1) Let vector $\mathbf{A}^{(i)} = (A_{i1}, A_{i2}, A_{i3})$. Now

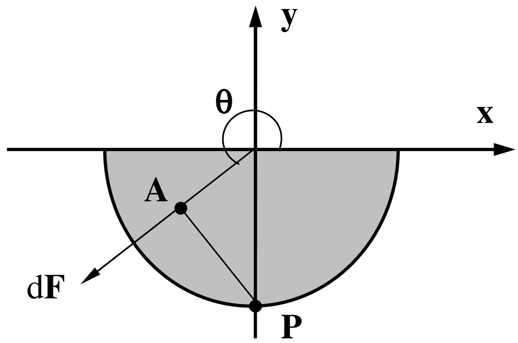
$$\int_S A_{ij}\, dS_j = \int_S \mathbf{A}^{(i)} \cdot d\mathbf{S} \overset{(1)}{=} \int_V \nabla \cdot \mathbf{A}^{(i)}\, dV = \int_V \frac{\partial A_{ij}}{\partial x_j}\, dV, \; \forall i.$$

(Version 2) Let's multiply tensor A_{ij} from left by a constant vector a_i, then we are left with vector $b_j = a_i A_{ij}$. For that

$$\begin{aligned} a_i \int_S A_{ij} dS_j &= \int_S a_i A_{ij} dS_j = \int_S b_j\, dS_j = \int_S \boldsymbol{b} \cdot d\boldsymbol{S} \\ &= \int_V \nabla \cdot \boldsymbol{b}\, dV = \int_V \frac{\partial (a_i A_{ij})}{\partial x_j}\, dV = a_i \int_V \frac{\partial A_{ij}}{\partial x_j}\, dV \end{aligned}$$

One must notice that there is a summation over i in the above relation, thus simply dividing by a_i is not allowed. Though, the above relation holds for every constant vector, thus we can first choice $a_i = \delta_{1i}$, then $a_i = \delta_{2i}$ and for the last $a_i = \delta_{3i}$, after which the relation $\int_S A_{ij} dS_j = \int_V \frac{\partial A_{ij}}{\partial x_j}\, dV$ is proved for all i.

2. A gutter is in the form of half a cylinder and is full of water (see figure).
a) Prove, by integrating surface forces, that the total force on the gutter is equal to the weight of water in the gutter.
b) Calculate the moment, about the lowest level of the gutter, of the surface forces on the half of the gutter on one side of this lowest line.
c) Calculate the force on one half of the gutter.
Solution:



We consider the case in the xy-plane and neglect the z-direction.
Surface element is $d\mathbf{S} = -ad\theta\hat{\mathbf{r}}$ (pointed from the solid into the fluid), where a is the radius of the gutter. The pressure is $p = p_0 - \rho g y = p_0 - \rho g a \sin\theta$ (pressure increases with decreasing y).

a) (Version 1) The force per surface element is found from $dF_i = \sigma_{ij} dS_j = -p dS_i$. The total force on the gutter is found by integrating over the surface S of the gutter

$$\mathbf{F} = -\int_S p\, d\mathbf{S} = -\int_V \nabla p\, dV = -\rho V g \hat{\mathbf{y}} = -M g \hat{\mathbf{y}},$$

which equals the weight of water in the gutter. In the integral above we have used the result of exercise 1 a).
(Version 2) The force per surface element ($d\boldsymbol{S} = dS\hat{\mathbf{n}} = dS\hat{\mathbf{r}}$ and $dS = ad\theta$) is found in a point (a, θ) from

$$\begin{aligned} d\boldsymbol{F} &= -pdS\hat{\mathbf{r}} = -\rho g r \sin\theta a d\theta(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}) \\ &= \rho g a^2 d\theta(-\sin\theta\cos\theta\hat{\mathbf{x}} - (\sin\theta)^2\hat{\mathbf{y}}) \end{aligned}$$

To get the total force $\boldsymbol{F}$, $d\boldsymbol{F}$ is integrated over the surface of the gutter:

$$\boldsymbol{F} = \int_\pi^{2\pi} d\boldsymbol{F} = \rho g a^2 \int_\pi^{2\pi} (-\cos\theta\sin\theta\hat{\mathbf{x}} - \sin^2\theta\hat{\mathbf{y}})d\theta = -\rho g a^2 \frac{\pi}{2}\hat{\mathbf{y}}.$$

Taking into account the depth l of the gutter the total force is

$$\boldsymbol{F} = -g\rho a^2 \frac{\pi}{2} l \hat{\mathbf{y}} = -gM\hat{\mathbf{y}}.$$

b) (Version 1) Next we calculate the moment about the lowest point, P, of the surface forces on the left side of the gutter. From the figure, distance $PA = -a\cos\theta$, and $|d\mathbf{F}| = -\rho g a^2 \sin\theta d\theta$. Thus the moment is

$$-\int_\pi^{3\pi/2} \rho g a^3 \sin\theta\cos\theta\, d\theta = \frac{1}{2}\rho g a^3.$$

(p_0 has been neglected as irrelevent.)
(Version 2) The moment $d\boldsymbol{T}$ has general definition: $d\boldsymbol{T} = \boldsymbol{r} \times d\boldsymbol{F}$. The vector $\boldsymbol{r}$ is the radius vector from the point P to the point where force $d\boldsymbol{F}$ acts.

$$\boldsymbol{r} = a\cos\theta\hat{\mathbf{x}} + (a + a\sin\theta)\hat{\mathbf{y}} \qquad d\boldsymbol{F} = \rho g a^2 d\theta(-\sin\theta\cos\theta\hat{\mathbf{x}} - \sin^2\theta\hat{\mathbf{y}})$$

$$\boldsymbol{r} \times d\boldsymbol{F} = \hat{\mathbf{z}}\rho g a^3 d\theta \sin\theta\cos\theta$$

The total moment with respect to the point P from the right hand side of the gutter:

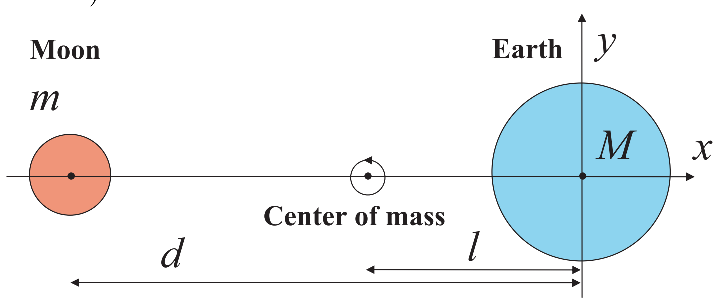
$$\boldsymbol{T} = \int_{\pi}^{3\pi/2} d\boldsymbol{T} = \rho g a^3 \int_{\pi}^{3\pi/2} \hat{\mathbf{z}}\sin\theta\cos\theta\, d\theta = \frac{1}{2}\rho g a^2 \hat{\mathbf{z}} \tag{3}$$

c) The force on the left half of the gutter is, (neglecting the air pressure p_0)

$$\mathbf{F} = -\int_S p\, d\mathbf{S} = \int_{\pi}^{3\pi/2} pa\, d\theta\hat{\mathbf{r}}$$

$$= -\rho g \int_{\pi}^{3\pi/2} a^2 \sin\theta(\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}})\, d\theta = -\rho g a^2 \left(\frac{1}{2}\hat{\mathbf{x}} + \frac{\pi}{4}\hat{\mathbf{y}}\right).$$

3. The starting point for studying tides is to consider the Earth and the Moon circulating around their center of mass with angular velocity Ω. Tides are caused by the effect of the Moon's gravitational potential $\phi_m = -\gamma m/r'$ (where r' is the distance from Moon's center) on the surface of the Earth.



M and m are the masses of the Earth and the Moon, respectively, and d is the distance between the centers of the Earth and the Moon. The rotation axis of the Earth-Moon system is perpendicular to the x–y plane.

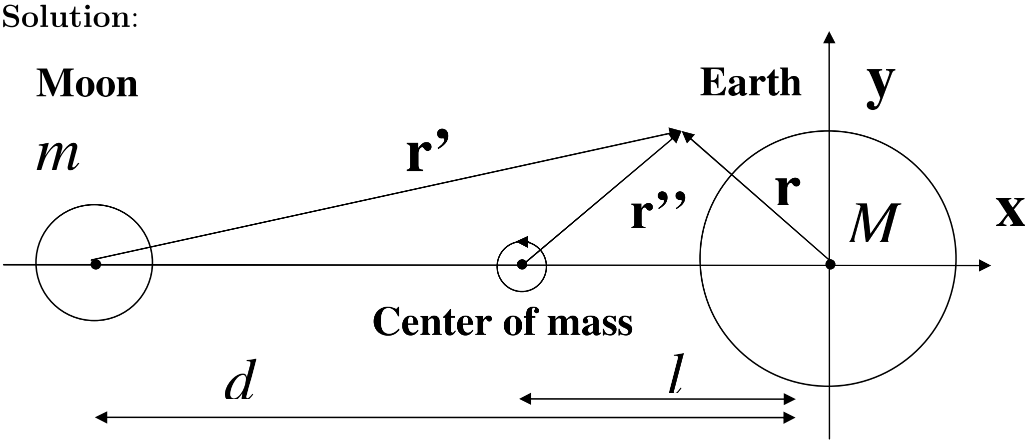
a) Determine the distance l of the center of mass from the center of the Earth.
b) Express the potential ϕ_m as a function of x, y and z.
c) By expanding ϕ_m in Taylor series up to second order in x/d, y/d and z/d, and neglecting all constant and higher-order terms show that

$$\phi_m = \frac{\gamma m}{d^2}x - \frac{\gamma m}{2d^3}(2x^2 - y^2 - z^2) \tag{4}$$

d) We now argue that the term linear in x in (4) causes the centripetal acceleration that keeps the Earth at constant distance from the center of mass. Show that this leads to the condition $\Omega^2 d^3 = \gamma(m + M)$.
e) Take into account also Earth's gravitational potential near the surface $\phi_e = gh$. (Here h is the height and the g can also be expressed as $g = \gamma M/R_e^2$, where R_e

is the radius of the Earth.) Using this together with the quadratic terms in (4), express the condition for the sea level in hydrostatic equilibrium, and calculate numerically the maximum height of the tide. (Warning: assuming hydrostatic equilibrium severely underestimates the tide near coastlines. Also other bodies, especially the Sun, contribute to tides.)

Solution:



a) The relations between vectors in the figure are

$$\begin{aligned} \mathbf{r}' &= d\,\mathbf{i} + \mathbf{r} = (d+x)\mathbf{i} + y\mathbf{j}, \\ \mathbf{r}'' &= l\mathbf{i} + \mathbf{r} = (l+x)\mathbf{i} + y\mathbf{j}. \end{aligned}$$

The center of gravity is at distance

$$l = \frac{M \cdot 0 + md}{M+m} = \frac{m}{m+M} d$$

from Earth ($l \sim$4565 km, i.e. it is actually inside Earth).

b) Writing r' in terms of x, y and z, we get

$$\begin{aligned} \phi_m &= -\frac{\gamma m}{r'} = -\frac{\gamma m}{\sqrt{(d+x)^2 + y^2 + z^2}} \\ &= -\frac{\gamma m}{d} \frac{1}{\sqrt{1 + 2x/d + x^2/d^2 + y^2/d^2 + z^2/d^2}} \end{aligned}$$

c) Using the expansion $(1+x)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \ldots$ we get the potential

$$\begin{aligned} \phi_m &= -\frac{\gamma m}{d}\left[1 - \frac{1}{2}\left(\frac{2x}{d} + \frac{x^2}{d^2} + \frac{y^2}{d^2} + \frac{z^2}{d^2}\right) + \frac{3}{8}\left(\frac{2x}{d} + \frac{x^2}{d^2} + \frac{y^2}{d^2} + \frac{z^2}{d^2}\right)^2\right] \\ &= -\frac{\gamma m}{d}\left[1 - \frac{2xd + x^2 + y^2 + z^2}{2d^2} + \frac{3}{8}\left(\frac{2x}{d}\right)^2\right] + O_3 \\ &= -\gamma m\left(\frac{1}{d} - \frac{x}{d^2} + \frac{2x^2 - y^2 - z^2}{2d^3}\right) + O_3. \end{aligned}$$

Neglecting the constant and O_3 terms, we get

$$\phi_m = \frac{\gamma m}{d^2}x - \frac{\gamma m}{2d^3}(2x^2 - y^2 - z^2).$$

d) Argue that the linear term in x in (4), in other words $\gamma mx/d^2$, causes the centripetal acceleration. The potentials are $\phi_m^{\text{linear}} = \gamma mx/d^2$ and $\phi_\Omega = -\frac{1}{2}\Omega^2 x^2$. The corresponding forces are

$$\begin{aligned}\mathbf{F}_m &= -\nabla\phi_m^{\text{linear}} = -\gamma m/d^2\mathbf{i}\\ \mathbf{F}_\Omega &= -\nabla\phi_\Omega = \Omega^2 x\mathbf{i}.\end{aligned}$$

The total force at the center of the Earth ($x = l$) is

$$\mathbf{F}_m + \mathbf{F}_\Omega = -\left(\frac{\gamma m}{d^2} - \Omega^2\frac{m}{m+M}d\right)\mathbf{i}.$$

This must vanish for Earth to stay in orbit, which leads to

$$\begin{aligned}\frac{\gamma m}{d^2} &= \Omega^2\frac{m}{m+M}d\\ \Rightarrow \Omega^2 d^3 &= \gamma(m+M).\end{aligned}$$

e) Earth's gravitational potential near the surface is $\phi_e = gh = \gamma Mh/R_e^2$, where R_e is the radius of the Earth. Taking into account the quadratic terms in ϕ_m, the total potential is

$$\begin{aligned}\phi &= \phi_e + \phi_m^{\text{quadratic}}\\ &= \frac{\gamma M}{R_e^2}h - \frac{\gamma m}{2d^3}(2x^2 - y^2 - z^2)\\ &= \frac{\gamma M}{R_e^2}h - \frac{\gamma m}{2d^3}(3x^2 - x^2 - y^2 - z^2)\\ &= \frac{\gamma M}{R_e^2}h - \frac{\gamma m}{2d^3}(3\cos^2\theta - 1)R_e^2\end{aligned}$$

where $r = R_e$ near the surface of the Earth. The potential at the maximum of the tide, for example $\theta = 0$ where $h = h_x$, and at the minimum of the tide, for example $\theta = \pi/2$ where $h = h_y$, is

$$\begin{aligned}\phi_x &= \frac{\gamma M}{R_e^2}h_x - \frac{\gamma m}{2d^3}2R_e\\ \phi_y &= \frac{\gamma M}{R_e^2}h_y - \frac{\gamma m}{2d^3}(-R_e^2).\end{aligned}$$

In case of hydrostatic equilibrium, $\phi =$ constant, and thus $\phi_x = \phi_y$. Then

$$\begin{aligned} \frac{\gamma M}{R_e^2} h_x - \frac{2R_e^2 \gamma m}{2d^3} &= \frac{\gamma M}{R_e^2} h_y + \frac{R_e^2 \gamma m}{2d^3} \\ \Rightarrow h_x - h_y &= \frac{3mR_e^4}{2d^3 M}. \end{aligned}$$

Using masses $m = 7.348 \times 10^{22}$ kg and $M = 5.974 \times 10^{24}$ kg for the Moon and the Earth, respectively, and the radius of Earth $R_e = 6371$ km and the distance of the Moon from the Earth $d = 384400$ km,

$$h_x - h_y \approx 0.5 \text{ m}.$$

The mean observed tide height is of order 8 m. The tides are actually a dynamic phenomenon, not static as assumed here. (For more information on tides, see http://scienceworld.wolfram.com/physics/Tide.html)