1. **Microscopic model of viscosity**
The viscosity of a gas can be estimated as $\mu = \frac{1}{3}\rho v \lambda$, where $v = \sqrt{3k_bT/m}$ is the average velocity of molecules and λ is the mean free path. Estimate μ numerically for air (use the results $\lambda = 570$ nm from exercise set 1, problem 5, and mass $m = 4.65 \cdot 10^{-26}$ kg for N_2) and compare with the measured value.
Solution: We use the values $T = 293$ K, $k_B = 1.381 \cdot 10^{-23}$ JK^{-1} nm, and $a = 150$ pm. The density can be written as $\rho = nm$, where n is the number density. From the problem 5 of exercise set 1, we have the formula for the mean free path

$$\lambda = \frac{1}{n\sigma},$$

so for number density $n = 1/(\lambda\sigma)$, where the cross section $\sigma = \pi a^2$. Thus we get

$$\mu = \frac{1}{3}\rho v \lambda = \frac{1}{3} nm \sqrt{\frac{3k_bT}{m}}\lambda = \sqrt{\frac{mk_BT}{3\sigma^2}} \approx 1.68 \cdot 10^{-4}\ \frac{\mathrm{kg}}{\mathrm{ms}}.$$

The measured value is $\mu = 1.8 \cdot 10^{-5}$ kg/(ms), factor 10 smaller than calculated above. Using the measured density $\rho = 1.2$ kg/m^3, and the given mean free path (calculated in room temperature $T = 293$ K), the result is $\mu = 1.16 \cdot 10^{-4}$.

2. **Viscous stress tensor**
   (a) The form of the stress tensor, assuming $\nabla \cdot \mathbf{v} = 0$,

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

is valid in cartesian coordinates. Applying the formulas given in appendix B of the lecture notes show that in plane polar coordinates the stress tensor takes the form

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}, \qquad \sigma_{\theta r} = \sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right).$$

   (b) The stream function

$$\Psi = U\left(r - \frac{a^2}{r}\right)\sin\theta$$

gives a model flow past a cylinder of radius a. Calculate the components of the viscous stress tensor σ'_{ij} in plane polar coordinates.

(c) Calculate the viscous force on the cylinder. Is it realistic?

Solution:

(a) Another way to express tensors is to understand them as products of vectors, this can be useful when changing coordinate system, for example. This procedure was applied in the appendix B of the lecture notes, such that

$$\sigma_{ij} = -p\delta_{ij} + \mu\left(\boldsymbol{v}\overleftarrow{\nabla} + \nabla\boldsymbol{v}\right)_{ij},$$

where the tensor $2e_{ij} = (\boldsymbol{v}\overleftarrow{\nabla} + \nabla\boldsymbol{v})_{ij}$ is in plane polar coordinates (see appendix)

$$(\boldsymbol{v}\overleftarrow{\nabla} + \nabla\boldsymbol{v})_{ij} = 2\frac{\partial v_r}{\partial r}\hat{\boldsymbol{r}}\hat{\boldsymbol{r}} + \left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right)(\hat{\boldsymbol{r}}\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{\theta}}\hat{\boldsymbol{r}}) + 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)\hat{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}.$$

Now, we simply write the answer by picking the corresponding components of the tensor

$$\sigma_{rr} = -p + 2\mu\frac{\partial v_r}{\partial r}, \qquad \sigma_{\theta r} = \sigma_{r\theta} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right),$$

$$\sigma_{\theta\theta} = -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right).$$

(b) In the problem 3 of the exercise 3, we had exactly the same stream function Ψ. From there we can take the velocity field:

$$v_r = \frac{1}{r}\frac{\partial \Psi}{\partial \theta} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \qquad v_\theta = -\frac{\partial \Psi}{\partial r} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$

Applying the expressions for the stress tensor calculated above, we get the viscous stress tensor σ'_{ij} through, $\sigma'_{ij} = \sigma_{ij} + p\delta_{ij}$:

$$\sigma'_{rr} = 2\mu\frac{\partial v_r}{\partial r} = 4U\mu a^2\frac{\cos\theta}{r^3},$$

$$\sigma'_{r\theta} = \sigma'_{\theta r} = \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r}\right) = 4U\mu a^2\frac{\sin\theta}{r^3},$$

$$\sigma'_{\theta\theta} = 2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right) = -4U\mu a^2\frac{\cos\theta}{r^3}.$$

(c) The i$^{\text{th}}$ component of the force is $dF_i = \sigma_{ij}dS_j = \sigma_{ij}n_j dS$. The cylinder radius is a, so the surface element is $d\boldsymbol{S} = ad\theta\hat{\boldsymbol{r}}$, i.e. $n_r = 1$ and $n_\theta = 0$ on the surface.
In the vector form, we can write $d\boldsymbol{F} = dF_i\hat{\mathbf{e}}_i$ and

$$\boldsymbol{F} = \int_S dF_i\hat{\mathbf{e}}_i = \int_S \sigma_{ij}n_j\hat{\mathbf{e}}_i\,dS = \int_S (\sigma_{rr}n_r + \sigma_{r\theta}n_\theta)\hat{\mathbf{r}} + (\sigma_{\theta r}n_r + \sigma_{\theta\theta}n_\theta)\hat{\boldsymbol{\theta}}\,dS.$$

Inserting $n_r = 1$ and $n_\theta = 0$ gives

$$\boldsymbol{F} = \int_S \sigma_{rr}\hat{\mathbf{r}} + \sigma_{\theta r}\hat{\boldsymbol{\theta}}\,dS = \int_0^{2\pi} (\sigma_{rr}\hat{\mathbf{r}} + \sigma_{\theta r}\hat{\boldsymbol{\theta}})\,ad\theta.$$

To be able to calculate the total force in vector form, we need to use cartesian unit vectors $\boldsymbol{i}$ and $\boldsymbol{j}$. We cannot use $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$, since they depend on θ, and would change during the integration. Inserting

$$\hat{\mathbf{r}} = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j}$$

into the integral gives

$$\boldsymbol{F} = \int_0^{2\pi} \left[(\sigma_{rr}\cos\theta - \sigma_{\theta r}\sin\theta)\boldsymbol{i} + (\sigma_{rr}\sin\theta + \sigma_{\theta r}\cos\theta)\boldsymbol{j}\right] a d\theta.$$

Taking $\sigma_{rr} = -p + \sigma'_{rr} = -p + 4\mu r^{-3}a^2U\cos\theta$ and $\sigma_{r\theta} = \sigma'_{r\theta} = 4\mu r^{-3}a^2U\sin\theta$ from the previous exercise gives

$$\boldsymbol{F} = 4\mu r^{-3}a^2U \int_0^{2\pi} \left[(\cos^2\theta - \sin^2\theta)\boldsymbol{i} + 2\sin\theta\cos\theta\boldsymbol{j}\right] a d\theta = 0.$$

The result is not realistic (d'Alembert's paradox). Surely, we feel a force if we stick a rod into a flowing river. The main reason, that our model flow is not realistic, is that the flow assumes wrong boundary condition at the surface of the cylinder. Clearly, the velocity does not vanish at the surface of the cylinder, schematized in Fig. **??**, discussed also in the lectures, and in Ex. 3.3. Later on the course, we will see that the drag forces on a body arise from the implications of the separation layer around the surface of the body.

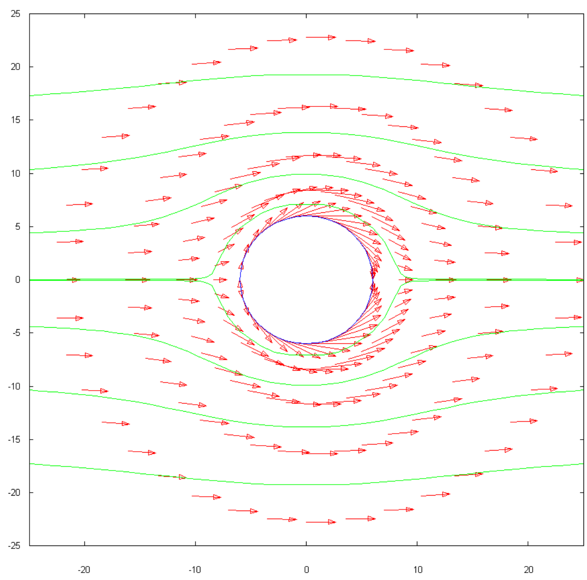

Figure 1: The velocity field (red arrows) near the cylinder (blue circle) and the stream-lines. Notice the non-vanishing velocity field at the surface of the cylinder.

3. **Reynold's number estimates**
Calculate the Reynolds number and comment the relative importance of inertial and viscous forces in the following cases.
Solution: The Reynold's number was defined in the lectures as:

$$\mathrm{Re} = \frac{\rho v a}{\mu} = \frac{v a}{\nu},$$

where ρ and μ denotes, respectively the density and the viscosity of the medium (kinematic viscosity $\nu = \mu/\rho$), whereas v and a denote, respectively, the characteristic velocity and scale of the object or the construction.

a) A swimmer's kick: $a = 50$ cm, $v = 30$ cm/s, $\nu_{\mathrm{water}} = 1.1 \cdot 10^{-6}$ m^2/s and Re $\sim 1.4 \cdot 10^5$. Inertial forces are the most important but viscosity may have affect though.

b) A bacterium in water: $a = 1$ μm, $v = 30$ μm/s, $\nu_{\mathrm{water}} = 1.1 \cdot 10^{-6}$ m^2/s and Re $\sim 2.7 \cdot 10^{-5}$. Opposite to the case of the swimmer's kick, inertial forces are negligible and viscosity dominates. As an implication bacteria uses different kind of swimming strategy, i.e. they may use rotating corkscrew-shaped swimming flagellum.

c) A river: $a = 10$ m, $v = 10$ cm/s, $\nu_{\mathrm{water}} = 1.1 \cdot 10^{-6}$ m^2/s and Re $\sim 9.1 \cdot 10^5$. Inertial forces dominate but viscosity may have affect though.

d) The climate: $a = 1000$ km, $v = 10$ m/s, $\nu_{\mathrm{air}} = 1.5 \cdot 10^{-5}$ m^2/s and Re $\sim 6.8 \cdot 10^{11}$. Inertial forces dominate.

e) A glacier: $a = 100$ m, $v = 1$ m/year, $\mu \sim 10^{10}$ kg/(ms), $\rho_{\mathrm{ice}} = 916.7$ kg/m^3 and Re $\sim 2.9 \cdot 10^{-13}$. Viscous forces dominate.

f) An accretion disk around a black hole: $a = 10^5$ m, $v = 10^7$ m/s, $\nu \sim 10^2$ m$^2/s$ and Re $\sim 10^{10}$. Inertial forces dominate although the value of the kinematic viscosity is huge.