1. **Plane Couette flow**
Consider fluid between parallel planes. The wall at $y = 0$ is fixed, and the wall at $y = a$ moves with steady speed V in its own plane. Solve the Navier-Stokes equations for the case $\rho =$ constant to show that a possible flow is

$$\mathbf{v} = \frac{Vy}{a}\mathbf{i}.$$

Calculate the stress on both walls.
Solution:
Modeling: An obvious choice for the velocity field of the fluid is $\boldsymbol{v} = \boldsymbol{v}(y) = U(y)\boldsymbol{i}$. The inner pressure of the fluid is assumed constant, thus $\nabla p = 0$. Also it is assumed that no volume forces are present: $\rho \boldsymbol{f} = 0$. The situation also seems to be static so that partial derivative of the velocity vanishes: $\partial \boldsymbol{v}/\partial t = 0$.

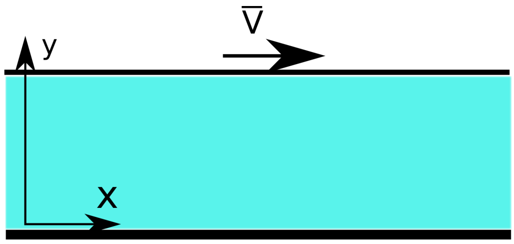

Figure 1: Schematics of the plane Couette flow: The lower boundary is fixed and the upper boundary moves with velocity V to right.

Boundary conditions: The upper plane, at $y = a$, is moving to the direction of $\boldsymbol{i}$ with velocity V. The boundary condition for velocity of the fluid $\boldsymbol{v}$ at $y = a$ is then

$$\boldsymbol{v}(y = a) = U(a)\boldsymbol{i} = V\boldsymbol{i}.$$

This condition says that the upper wall sees the fluid at rest. The second boundary condition says that also the lower wall sees the fluid at rest:

$$\boldsymbol{v}(y = 0) = 0.$$

Solution: The equation of motion, that is, the Navier-Stokes equation, stands as

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \boldsymbol{f} - \nabla p + \mu \nabla^2 \boldsymbol{v}$$

and reduces to the form

$$\rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \mu \nabla^2 \boldsymbol{v}$$

from which one sees that the left hand side also vanishes since $\boldsymbol{v}\cdot\nabla\boldsymbol{v} = U(y)\partial U(y)/\partial x = 0$. The equation

$$0 = \frac{\partial^2 U(y)}{\partial y^2}$$

is solved with function $U(y) = Dy + E$. The second boundary condition reduces the solution to the form $U(y) = Dy$ and the first boundary condition further to the form

$$\boldsymbol{v}(y) = U(y)\boldsymbol{i} = V\frac{y}{a}\boldsymbol{i}.$$

Stress on the walls: The force per area unit, stress, is calculated from the stress tensor

$$\frac{dF_i}{dS} = \sigma_{ij}n_j,$$

where n_j is the normal vector of the surface. The stress tensor

$$\begin{aligned}\sigma_{ij} &= -p\delta_{ij} + \sigma'_{ij} \\ &= -p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{v_k}{x_k}\right) + K\delta_{ij}\frac{v_k}{x_k},\end{aligned}$$

which is just

$$\sigma_{ij} = -p\delta_{ij} + \mu\frac{V}{a}\left(\delta_{i2}\delta_{j1} + \delta_{i1}\delta_{j2}\right).$$

The surface normal points towards the fluid: at the upper to negative y direction $\hat{\mathbf{n}}^{\uparrow} = -\boldsymbol{j}$ and at the lower plane to positive y direction $\hat{\mathbf{n}}^{\downarrow} = \boldsymbol{j}$. Thus the stress is

$$\frac{dF_x}{dS} = \mp\sigma_{xy}n_y = \mp\mu\frac{V}{a} \qquad\qquad \frac{dF_y}{dS} = \mp\sigma_{yy}n_y = \pm p$$

(upper sign for upper plane).

2. **Flow down a slope** (solving this problem gives double points)
A liquid of constant density flows down a plane which slopes at angle α to the horizontal, as indicated in the figure below. The free surface of the liquid is at a uniform distance from the plane, has pressure p_0 and no shear stress. For this flow you need to keep the gravitational field in the Navier-Stokes equation, as it is now dynamically active. Set up and solve equations for $U(y)$, and verify that the forces on a length l of the fluid layer are in equilibrium.
Solution:
Modeling: We have an equilibrium situation, where the fluid velocity $\boldsymbol{v} = U(y)\boldsymbol{i}$ and it does not change in time. The gravitational acceleration in the chosen coordinate system is

$$\boldsymbol{g} = g(\boldsymbol{i}\sin\alpha - \boldsymbol{j}\cos\alpha),$$

whose x-component is responsible for the motion of the fluid, while the y-component generates the hydrostatic pressure as we'll see soon. It is not applied external pressure

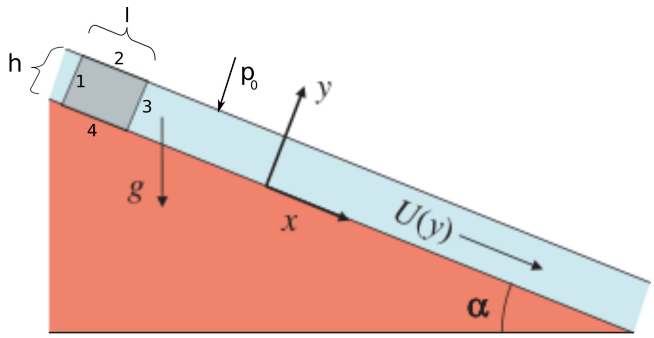

Figure 2: Schematics of the flow down a slope

gradient over the free ends of the liquid, so the pressure varies only in the y-direction such that $p(h) = p_0$, where h is the height of the fluid layer.
Solution of the Navier-Stokes equation: Using $\boldsymbol{f} = \boldsymbol{g}$, the full Navier-Stokes equation

$$\rho \frac{D\boldsymbol{v}}{Dt} = \rho \boldsymbol{f} - \nabla p + \mu \nabla^2 \boldsymbol{v}$$

reduces to the equilibrium condition in tha y- and x direction

$$-\rho g \cos\alpha = \frac{\partial p}{\partial y}, \tag{1}$$

$$0 = \rho g \sin\alpha - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 U(y)}{\partial y^2} \tag{2}$$

Now there is no pressure gradient in x-direction, so $\partial p/\partial x = 0$. The pressure on the free surface of the fluid is just the air pressure $p(y = h) = p_0$. Integrating eq. (1), we get

$$p(y) = -\rho g \cos\alpha y + a,$$

and from the boundary condition $p(h) = p_0$ we get $a = p_0 + \rho g \cos\alpha h$. The pressure is

$$p = \rho g \cos\alpha (h - y) + p_0. \tag{3}$$

From eq. (2), we now get

$$-\rho g \sin\alpha = \mu \frac{\partial^2 U}{\partial y^2} \Rightarrow U(y) = -\frac{\rho g \sin\alpha}{2\mu} y^2 + by + c,$$

where b and c are integration constants, which can be identified from the boundary conditions. On the solid surface the fluid velocity is zero, $U(0) = 0$, implying $c = 0$. On the free surface there is no shear stress apart from the pressure p_0, so $\sigma_{xy} = 0$ at $y = h$, from which we get the second condition for the velocity. From the expression

$$\sigma_{xy} = \mu \left(\frac{\partial U}{\partial y} \right) = -\rho g \sin\alpha y + \mu b$$

we find at $y = h$ that

$$b = \frac{-\rho g \sin \alpha h}{\mu}.$$

Finally:

$$U(y) = \frac{\rho g \sin \alpha}{2\mu}(2hy - y^2).$$

The velocity field $U(y)$ is analogous to the fluid flow between walls, that is, to the example 7.1 of the lectures. Now, the velcity profile is parabolic, $\rho g \sin \alpha$ acts as an pressure gradient and the smaller viscosity μ the faster flow on the tilted plane.
Force balance: Next, we will verify that the forces on a portion of fluid, sketched in Fig. 2, are in equilibrium. We consider a volume $V = lwh$, where l is the length, w the width, and h the height of the fluid. The gravitation is the only volume force acting to this fluid element:

$$\boldsymbol{F}^{(V)} = V\boldsymbol{f} = lwh\boldsymbol{g}.$$

The second class of forces are the shear (surface) forces. Thus, we use the equation $dF_i = \sigma_{ij} n_j \, dS$ for the force *on the fluid element* due to the stress tensor. The normal vectors point out of the fluid element. Now, the components of the stress tensor are

$$\sigma_{xy} = \rho g \sin \alpha (h - y) \qquad \sigma_{xx} = \sigma_{yy} = -p = -\rho g \cos \alpha (h - y) - p_0$$

We first calculate the surface forces acting on the boundaries (1) and (3) of the fluid element, shown in Fig. 2.
First for the boundary (1), the surface element is now $\mathrm{d}S_j = -w\mathrm{d}y \, \delta_{xj}$ and thus the force

$$\boldsymbol{F}^{(1)} = \int_{S_1} \boldsymbol{i}\sigma_{xk}\mathrm{d}S_k + \int_{S_1} \boldsymbol{j}\sigma_{yk}\mathrm{d}S_k = \int_0^h \boldsymbol{i}\sigma_{xx}(y)(-w\mathrm{d}y) + \int_0^h \boldsymbol{j}\sigma_{yx}(y)(-w\mathrm{d}y).$$

For the boundary (3), the surface element points to the opposite direction ($\mathrm{d}S_j = w\mathrm{d}y \, \delta_{xj}$) but otherwise the expression for the force is the same, that is, $\boldsymbol{F}^{(3)} = -\boldsymbol{F}^{(1)}$. At the topmost surface, the shear stress vanishes ($\sigma_{xy} = 0$) and it acts only the air pressure

$$\boldsymbol{F}^{(2)} = -lwp_0\boldsymbol{j}.$$

The force on the fluid volume through the solid surface (4) is

$$\boldsymbol{F}^{(4)} = \boldsymbol{j}\sigma_{yy}(y = 0)(-lw) + \boldsymbol{i}\sigma_{xy}(y = 0)(-lw) = lw(p_0 + hg\rho \cos \alpha)\boldsymbol{j} - lw\rho g \sin \alpha \boldsymbol{i}$$

The total force acting through the surfaces is sum over all the surface forces yielding

$$\boldsymbol{F}^{(S)} = \sum_{k=1}^{4} \boldsymbol{F}^{(k)} = lwh\rho(-\sin \alpha g\boldsymbol{i} + \cos \alpha g\boldsymbol{j}) = -lwh\rho\boldsymbol{g}.$$

Finally, we see that the volume forces are balanced by the surface forces

$$\boldsymbol{F} = \boldsymbol{F}^{(V)} + \boldsymbol{F}^{(S)} = 0$$

Thus, the flow is static.