1. **Dimensioless Euler's equation**
The Euler equation for incompressible flow in a rotating system was given in the form

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} + 2\rho\boldsymbol{\Omega}\times\boldsymbol{v} = -\boldsymbol{\nabla}p$$

in the lectures. Write this in a dimensionless form.
Solution: Following the lectures, let's choose the dimensions

$$t = t_0 T \qquad \boldsymbol{\nabla} = \frac{1}{x_0}\boldsymbol{\nabla}_R \qquad \boldsymbol{v} = v_0\boldsymbol{V} = \frac{x_0}{t_0}\boldsymbol{V} \qquad \boldsymbol{\Omega} = \frac{1}{t_0}\widetilde{\boldsymbol{\Omega}} \qquad p = p_0 P = \rho v_0^2 P$$

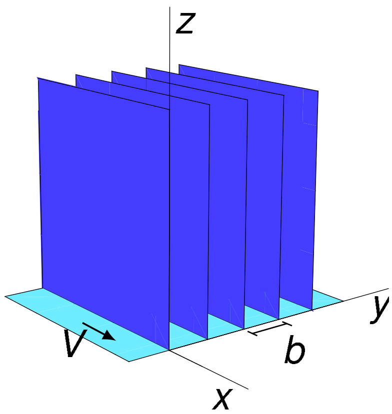
and density ρ is a constant. Notice the connected $x_0 = t_0 v_0$ and $p_0 = \rho v_0^2$. With these Euler equation reads as

$$\rho\frac{v_0}{t_0}\frac{\partial \boldsymbol{V}}{\partial T} + \rho\frac{v_0^2}{x_0}\boldsymbol{V}\cdot\boldsymbol{\nabla}_R\boldsymbol{V} + 2\rho\frac{v_0}{t_0}\widetilde{\boldsymbol{\Omega}}\times\boldsymbol{V} = -\frac{p_0}{x_0}\boldsymbol{\nabla}_R P$$
$$\rho\frac{v_0}{t_0}\left(\frac{\partial \boldsymbol{V}}{\partial T} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_R\boldsymbol{V} + 2\widetilde{\boldsymbol{\Omega}}\times\boldsymbol{V}\right) = -\frac{p_0}{x_0}\boldsymbol{\nabla}_R P$$
$$\frac{\partial \boldsymbol{V}}{\partial T} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_R\boldsymbol{V} + 2\widetilde{\boldsymbol{\Omega}}\times\boldsymbol{V} = -\frac{p_0}{\rho v_0^2}\boldsymbol{\nabla}_R P$$

and as the unit of pressure were chosen to be ρv_0^2. Thus, we have the dimensionless Euler equation

$$\frac{\partial \boldsymbol{V}}{\partial T} + \boldsymbol{V}\cdot\boldsymbol{\nabla}_R\boldsymbol{V} + 2\widetilde{\boldsymbol{\Omega}}\times\boldsymbol{V} = -\boldsymbol{\nabla}_R P.$$

2. **Paintbrush** (solving this problem gives double points)
Consider a simple model of a paintbrush consisting of parallel planes with spacing b and normal $\boldsymbol{j}$.



For convenience, assume the wall in the $x - y$ plane is moving with constant velocity $V\boldsymbol{i}$ and that the brush is stationary.
Determine the velocity of the paint between the brush planes assuming the form $\boldsymbol{v} = U(y,z)\boldsymbol{i}$. (Use the method of separation of variables.)
Calculate the total paint flow $Q = \int_0^b dy \int_0^\infty dz\, U(y,z)$ between two planes. Based on this deduce how thick is the layer of paint left on the wall.
[Answer: $Q = \frac{8Vb^2}{\pi^3}\sum_{n=1}^{\infty}(2n-1)^{-3} \approx 0.27Vb^2$.]

Solution:
Recap of the method of separtion of variables: Having $\boldsymbol{v} = U(y,z)\boldsymbol{i}$, we separate variables by setting $U(y,z) = Y(y)Z(z)$. The flow is now steady, that is, $\partial v/\partial t = 0$. The velocity is in the x-direction, but independent on x, that is why $v_k(\partial v_i/\partial x_k) = 0$ for all i, and $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v} = 0$. Also, there is no pressure gradient, $\nabla p = 0$, so the Navier-Stokes equation simply becomes

$$\nabla^2 \boldsymbol{v} = 0.$$

The x-component of this is

$$\nabla^2 U(y,z) = \frac{\partial^2 Y}{\partial y^2} Z(z) + Y(y)\frac{\partial^2 Z}{\partial z^2} = 0,$$

or

$$-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}.$$

The left side depends on y only and the right side on z only, thus, they must both be constant, say C. The velocity must vanish when $z \to \infty$, so the appropriate solution of $\partial^2 Z/\partial z^2 = cZ$ is

$$Z = Ae^{-kz},$$

where $k^2 = C$, and $k > 0$. Correspondingly $\partial^2 Y/\partial y^2 = -cY$ is solved by

$$Y = B\cos ky + C\sin ky.$$

The boundary conditions for the fluid flow are as usual: $U(y,0) = V$, $U(y,z) = 0$ when $z \to \infty$ (which we already used), and $U(mb,z) = 0$, where $m = 0,1,2,\ldots$. Note that these two conditions are in contradiction at points where both $z = 0$ and $y = mb$, since then the velocity should be V and 0 at the same time. We can relax the first condition by demanding $U(y \neq mb, 0) = V$. In terms of Y and Z we thus have $Y(mb) = 0$ and $Z(0) = V$.
The complete solution: Now, we see that $B = 0$, and from condition $Y(b) = 0$ we have $C\sin kb = 0$, so $k = n\pi/b$, where $n = 1,2,\ldots$. We write the solution corresponding to a single value of n as $U_n = A_n C_n e^{-k_n z}\sin k_n y$, where $k_n = n\pi/b$. The full solution of $U(y,z)$ can now be written as the sum

$$U(y,z) = \sum_{n=1}^{\infty} D_n e^{-k_n z}\sin k_n y,$$

where $D_n = A_n C_n$. The constants D_n can be found at $z = 0$ by multiplying the equation

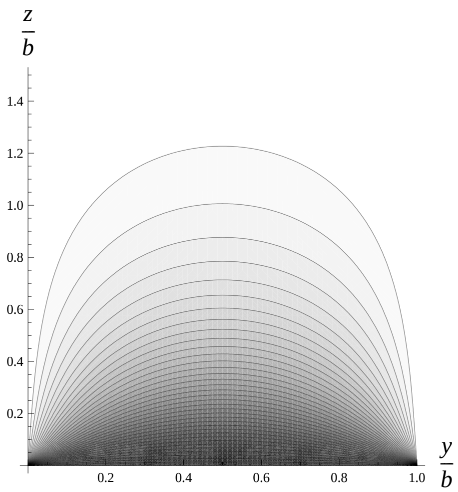
$$U(y,0) = V = \sum_{n=1}^{\infty} D_n \sin k_n y$$

by $\sin k_m y$ and integrating over y. We find, by using the orthogonality of sines,

$$D_n = \frac{2V}{b}\int_0^b \sin k_n y dy = -\frac{2V}{b}\frac{1}{k_n}\Big/_0^b \cos k_n y = -\frac{2V}{n\pi}((-1)^n - 1).$$

This is $4V/(n\pi)$ for any odd n, and zero for an even n. Thus, the full solution for the velocity flow is

$$U(y,z) = \frac{4V}{\pi} \sum_{n=1} \frac{1}{2n-1} e^{-zk_{2n-1}} \sin k_{2n-1} y. \tag{1}$$



The velocity profile of the fluid (1). The horizontal axis is the y-axis, and z is in the vertical direction. The darkness of the color indicates the magnitude of velocity, black being the largest. The velocity $U(y,z)$ of the fluid drops exponentially with increasing z, and is zero at the boundaries $y = 0$, $y = b$.

The total paint flow Q between the two planes is

$$Q = \int_0^b dy \int_0^\infty dz\, U(y,z) = \sum_{n=1}^\infty \frac{b}{2V} D_{2n-1}^2 \frac{1}{k_{2n-1}}$$

$$= \sum_{n=1}^\infty \frac{b}{2V} 16 \frac{V^2}{(2n-1)^2\pi^2} \frac{b}{(2n-1)\pi} = \frac{8b^2 V}{\pi^3} \sum_{n=1}^\infty \frac{1}{(2n-1)^3}.$$

We can write

$$\sum_{n=1}^\infty \frac{1}{(n-1)^3} = \sum_{n=1}^\infty \frac{1}{n^3} - \sum_{n=1}^\infty \frac{1}{(2n)^3} = \frac{7}{8} \sum_{n=1}^\infty \frac{1}{n^3}.$$

Now, we are amazed by our good luck, as we immediately notice that the sum $\sum_{n=1}^\infty \frac{1}{n^3}$ is, in fact, nothing but the Riemann zeta-function at 3,

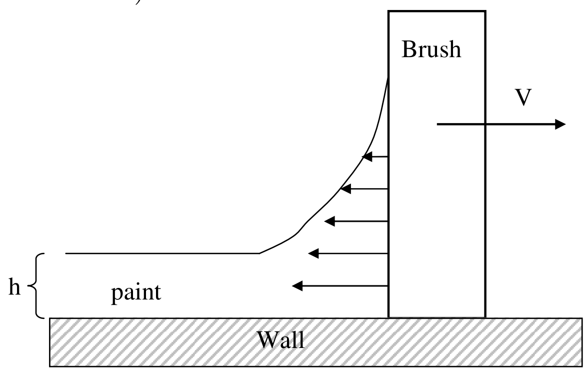
$$\zeta(3) = \sum_{n=1}^\infty \frac{1}{n^3} \approx 1.202.$$

This gives that the total flow $Q = \frac{7\zeta(3)}{\pi^3} b^2 V \approx 0.27 b^2 V$.

Thickness of the paint layer: Now, as the paint brush moves forward, it leaves behind it a layer of paint. Immediately behind the brush, there may be some complicated behavior of the paint, but far from it the paint forms a layer of uniform thickness h, see figure below. Considering a section of width b of the paint layer, corresponding to the slit between two planes of the brush, we see that the volume of this section increases by hbV as the brush moves forward. This amount must be equal to the paint flow Q, so we get

$$h = \frac{Q}{bV} = \frac{7b}{\pi^3} \zeta(3) \approx 0.27b.$$

We see that the thickness of the paint layer depends only on the distance b between the planes on the brush (in this approximation). In real world, the viscosity of the fluid may affect the boundary conditions between the brush planes and the paint, and between the wall and the paint. Also, the thermodynamic properties of the paint affect the form it assumes after it has left the brush. The velocity of the brush, on the other hand, affects the dynamics of the fluid flow through the Reynolds number (the fluid may become turbulent).



A schematic view of the situation. We have now chosen the coordinate system so that the brush moves to the right. The height of the fluid column just behind the brush is much higher than the final fluid layer thickness h far from the wire. However, the velocity of the fluid drops exponentially with increasing z, as indicated in the figure.

3. **Oscillating plane**
The plane $y = 0$ oscillates transversally with velocity $\boldsymbol{i}V\cos(\omega t)$. Show that the velocity of fluid $\boldsymbol{v} = U(y,t)\boldsymbol{i}$ above the plane ($y > 0$) has the form

$$U(y,t) = \Re[Ve^{i\omega t-(1+i)y/\delta}],$$

where $\delta = \sqrt{2\nu/\omega}$, i is the imaginary unit ($i^2 = -1$) and $\Re$ means the real part. Calculate the real part and discuss its form. Why is δ called "penetration depth"?

[Hint: Use ansatz $U = Ve^{i\omega t-ky}$ to solve the Navier-Stokes equations. This gives a complex solution but the real part of this corresponds to the physical solution.]

Solution: This exercise is somewhat similar to the boundary layer calculation in the beginning of the lecture note chapter 7.4. There is now at $y = 0$ a plane oscillating in $\boldsymbol{i}$ direction as $\cos\omega t$. The velocity of the plane is

$$V(t) = V\cos\omega t.$$

Situation is uniform in x-direction and thus velocity alternates only in y-direction. Naturally velocity field is time dependent. We write then

$$\boldsymbol{v} = U(y,t)\boldsymbol{i}$$

and deduce approriate boundary conditions

$$U(0,t) = V(t), \qquad\qquad U(\infty,t) = 0. \tag{2}$$

The full Navier-Stokes

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}\boldsymbol{v} = -\nabla p + \mu\nabla^2\boldsymbol{v}$$

reduces now to form

$$\frac{\partial U(y,t)}{\partial t} = \nu \frac{\partial^2 U(y,t)}{\partial y^2} \tag{3}$$

which we are going to solve with the ansatz

$$U(y,t) = \mathrm{Re}[V e^{i\omega t - ky}].$$

Let's first check that the ansatz is resonable in the sense of boundary conditions. First at the plate $y = 0$:

$$U(0,t) = \mathrm{Re}[V e^{i\omega t}] = V \cos \omega t \qquad\qquad \text{OK}$$

and far from the moving plate

$$\lim_{Y\to\infty} U(Y,t) = \lim_{Y\to\infty} \mathrm{Re}[V e^{i\omega t - kY}] = \lim_{Y\to\infty} \mathrm{Re}[V e^{i\omega t - i\mathrm{Im}(k)Y}] e^{-\mathrm{Re}(k)Y}.$$

The boundary condition $U(\infty, t) = 0$ is satisfied if $\mathrm{Re}(k) > 0$, with this in our mind we proceed further. By plugging the ansatz to the reduced Navier-Stokes (3), one arrives with equation

$$k^2 = i\omega/\nu$$

and with notation $k = \mathrm{Re}(k) + \mathrm{iIm}(k) = k_r + ik_i$ it is rewritten in the form of

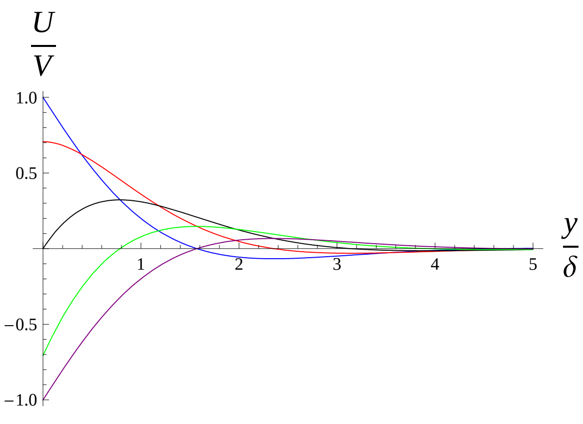
$$k_r^2 - k_i^2 + i2k_r k_i = i\omega/\nu$$

having solutions $k_r = \pm\sqrt{\frac{\omega}{2\nu}}, k_i = \pm\sqrt{\frac{\omega}{2\nu}}$ but the positive sign is only possibility due to the second boundary condition. The solution of the problem reads now

$$U(y,t) = \mathrm{Re}[V e^{i\omega t - (1+i)y\sqrt{\omega/2\nu}}] = V \exp\left(-\frac{y}{\sqrt{2\nu/\omega}}\right) \cos\left(\omega t - \frac{y}{\sqrt{2\nu/\omega}}\right).$$

Figure here illustrates this function. It is seen that solution oscillates both in time t and space y. The oscillation amplitude is damped by exponential term which goes to its e^{th} part when y increses by length of $\sqrt{2\nu/\omega}$. The $\delta = \sqrt{2\nu/\omega}$ sets the scale how the motion of the plate penetrates in the fluid. At the distance of a couple of penetration depths from the plate the fluid stands still.

Penetration depth is inverse propotional to the radial frequency of the oscillation, the faster oscillation the thinner boundary layer. In other words, low frequency disturbancies penetrate deeper into the fluid. Depth δ is propotional to the kinematic viscosity ν of the fluid, which is natural since the viscosity is responsible for the whole diffusion phenomenon. In low viscosity fluids, the boundary layer is thin.



The figure shows $U(y,t)$ plotted at five different time instances: $t = 0$ (blue), $\omega t = \pi/4$ (red), $\omega t = \pi/2$ (black), $\omega t = 3\pi/4$ (green), and $\omega t = \pi$ (purple)