1. **Transient flow between parallel planes** (double points)
Fluid is at rest in a long channel with rigid walls $y = \pm a$ when a pressure gradient $-G$ is suddenly imposed at $t = 0$.
   a) Show that the velocity $U(y,t)\boldsymbol{i}$ satisfies the equation
$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial y^2} + \frac{G}{\rho}$$
for $t > 0$, and state the boundary and initial conditions for this flow.
   b) As $t \to \infty$ we expect to get the flow appropriate for a pressure gradient in a channel $U_1(y) = \frac{G}{2\mu}(a^2 - y^2)$ so seek a solution in the form $U(y,t) = U_1(y) + V(y,t)$ what equation and boundary values does V satisfy?
   c) Show that $V(y,t)$ may be found by separation of variables. How long does it take for the flow U_1 to be established? Explain this answer physically.

Solution:

a) Since the velocity field is of the form of $\boldsymbol{v} = U(y,t)\boldsymbol{i}$, the term $(\boldsymbol{v} \cdot \nabla)\boldsymbol{v}$ vanishes similarly as in the previous exercises. Let us, in addition to that, assume that there is no dynamical forces $\boldsymbol{f}$ present. The Navier-Stokes equation is reduced to the desired form:
$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \nabla \boldsymbol{v} = \rho \boldsymbol{f} - \nabla p + \mu \nabla^2 \boldsymbol{v} \quad \Rightarrow$$
$$\frac{\partial U}{\partial t} = \frac{G}{\rho} + \nu \frac{\partial^2 U}{\partial y^2}. \tag{1}$$
Initial condition: At $t = 0$ the fluid is at rest, imposing $U(y, t = 0) = 0$.
Boundary condition: At the rigid walls $y = \pm a$, the velocity must vanish, thus $U(y = \pm a, t) = 0$.

b) Let us first consider the initial and boundary conditions for $V(y,t) = U(y,t) - U_1(y)$. At the initial time $t = 0$,
$$V(y, t = 0) = U(y, t = 0) - U_1(y) = -U_1(y). \tag{2}$$
And for the boundaries $y = \pm a$,
$$V(y = \pm a, t) = U(y = \pm a, t) - U_1(y) = 0. \tag{3}$$
In addition to these, we demand that $\lim_{t\to\infty} U(y,t) = U_1(y)$, which means that
$$\lim_{t\to\infty} V(y,t) = 0. \tag{4}$$

By plugging $U(y,t) = V(y,t) + \frac{G}{2\mu}(a^2 - y^2)$ into the actual Navier-Stokes equation, we get the equation

$$\frac{\partial V}{\partial T} = \nu \frac{\partial^2 V}{\partial t^2}. \tag{5}$$

c) By expressing V as a product $V(y,t) = Y(y)T(t)$, Eq. (5) becomes separated:

$$\frac{1}{\nu T(t)} \frac{\partial T(t)}{\partial t} = \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2}$$

This is split to two equations

$$T'(t) = -k^2 \nu T(t), \qquad Y''(y) - k^2 Y(y) = 0.$$

The choice $K = -k^2$ for the common coefficient guarantees that the solution for the $T(t) \to 0$ when $t \to \infty$, this is, Eq. (4). A solution for these equations is

$$T(t) = Ae^{-k^2 \nu t}, \qquad Y(y) = B\cos(ky) + C\sin(ky).$$

The consideration of the boundary condition (3) at $y = \pm a$ implies that $C = 0$ and $k_n = \pi(2n+1)/2a$, $n = 0, 1, 2, \ldots$. Thus, the full solution is superposition of all possible solutions:

$$V(y,t) = \sum_{n=0} B_n e^{-k_n^2 \nu t} \cos(k_n y). \tag{6}$$

The coefficients B_n are solved from the initial condition $V(y,0) = -U_1(y)$ by using the orthogonality

$$\int_{-a}^{a} \cos(k_m a)\cos(k_n a) \mathrm{d}y = \delta_{mn} a.$$

So we get by multiplying the initial condition $V(y,0) = -U_1(y)$ by $\cos(k_m y)$ and integrating that

$$B_m = \frac{1}{a} \int_{-a}^{a} -\frac{G}{2\mu}(a^2 - y^2)\cos(k_m y)\mathrm{d}y = (-1)^{m+1} \frac{2G}{\mu} k_m^{-3}.$$

Now, we write the the complete solution

$$V(y,t) = \frac{2G}{\mu} \sum_{n=0}^{\infty} \frac{(-1)^{m+1}}{k_m^3} e^{-k_n^2 \nu t} \cos(k_m y),$$

which is visualized in Fig. 1.

Transient time: The function $V(y,t)$ reduces roughly to its e^{th} part in time $\tau = 1/k_0^2 \nu$. The other factors $e^{-k_n^2 \nu t}$ vanish faster than the zeroth one ($n = 0$). Thus, the transient time is

$$\tau = \frac{1}{k_0^2 \nu} = \frac{4}{\pi^2} \frac{a^2}{\nu}.$$

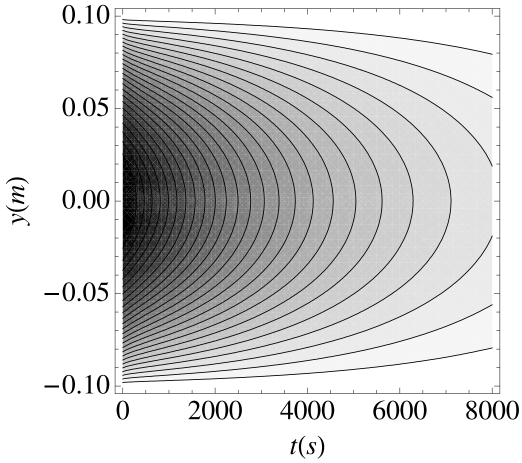
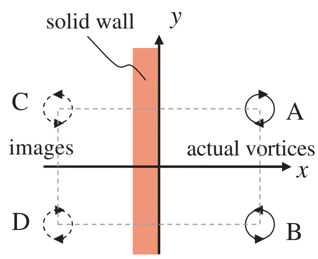


Figure 1: The flow $V(y,t)$ visualized. The darker color the larger is the velocity $-V(y,t)$.

Intuitively, the transient time is inversely proportional to viscosity. But quite a surprisingly, the time does not depend on the pressure gradient G but it depends on the separation $2a$ of the planes. But this is in agreement with the discussion in the lectures: The thickness of the separation layer is $d \approx \sqrt{\nu\tau}$. Now, one can, for example, think that the flow $V(y,t)$ represents a imaginary flow generated in between the planes such that the planes are moving to direction $-\boldsymbol{i}$ before time $t = 0$, but the planes stop moving at $t = 0$. The information of the stopping diffuses from both planes obeying the thickness relation $d \approx \sqrt{\nu\tau}$.

2. **Vortex pair near a wall**
Consider a pair of vortices, A and B, of circulations $-\kappa$ and κ, respectively, approaching a wall. The boundary condition for the normal component of the velocity at the wall, $v_x(0,y) = 0$, can be satisfied by adding two "image vortices" C and D, with circulations κ and $-\kappa$, respectively, behind the wall.



   a) Calculate the velocity at A induced by vortices B, C and D.
   b) Formulate a differential equation for the path of vortex A.
   c) Show that its solution is $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{x_0^2} + \frac{1}{y_0^2}$, and sketch the trajectory.

Solution:

   a) As derived in the lectures, the line vortex (at origin) with circulation κ has the the potential $\psi = -\frac{\kappa}{2\pi}\ln\frac{r}{a}$, and velocity $v_\theta = \frac{\kappa}{2\pi r}$, where r is the distance from the vortex line. Now, as seen from the figure, vortices C and B have circulation in the positive direction (counter-clockwise), while D and A in the negative direction (clockwise). If the vortex A is at point (x,y) (at the moment t), then B is at $(x,-y)$, C is at $(-x,y)$, and D is at $(-x,-y)$.

The distance from B to A is $2y$, so the velocity due to vortex B at A is

$$\boldsymbol{v}_B = -\frac{\kappa}{4\pi y}\boldsymbol{i}.$$

($\boldsymbol{v}_B$ is in the direction of $\hat{\boldsymbol{\theta}}'$ in a coordinate system where the origin is at B. Now, looking from B, the point A is at $\theta' = \pi/2$, and $\hat{\boldsymbol{\theta}}' = -\sin\theta'\boldsymbol{i} + \cos\theta'\boldsymbol{j} = -\boldsymbol{i}$. Similar reasoning is used for other points.)
The distance from C to A is $2x$, so the velocity due to vortex C at A is

$$\boldsymbol{v}_C = \frac{\kappa}{4\pi x}\boldsymbol{j}.$$

The distance from D to A is $2r$, where $r = \sqrt{x^2 + y^2}$, and the velocity $\boldsymbol{v}_D$ due to vortex D at A is in the $-\hat{\boldsymbol{\theta}}$-direction. Using $\hat{\boldsymbol{\theta}} = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} = -\frac{y}{r}\boldsymbol{i} + \frac{x}{r}\boldsymbol{j}$, we get

$$\boldsymbol{v}_D = \frac{\kappa}{4\pi}\left(\frac{y}{r^2}\boldsymbol{i} - \frac{x}{r^2}\boldsymbol{j}\right).$$

The total velocity at A is then

$$\begin{aligned}\boldsymbol{v}(x, y) = \boldsymbol{v}_B + \boldsymbol{v}_C + \boldsymbol{v}_D &= \frac{\kappa}{4\pi}\left[-\left(\frac{1}{y} - \frac{y}{r^2}\right)\boldsymbol{i} + \left(\frac{1}{x} - \frac{x}{r^2}\right)\boldsymbol{j}\right] \\ &= \frac{C}{2}\left(-\frac{x^2}{yr^2}\boldsymbol{i} + \frac{y^2}{xr^2}\boldsymbol{j}\right). \qquad (7)\end{aligned}$$

b) The differential equation for the motion of vortex A is obtained from the fluid velocity $\boldsymbol{v}$ Eq. (7) at the point A as

$$\begin{aligned}v_x = \frac{dx}{dt} &= -\frac{\kappa}{4\pi}\frac{x^2}{yr^2}, \\ v_y = \frac{dy}{dt} &= \frac{\kappa}{4\pi}\frac{y^2}{xr^2},\end{aligned}$$

from which we get the differential equation:

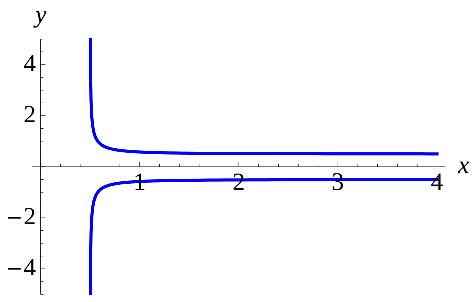
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{y^3}{x^3}.$$

c)

The equation is solved by separating the variables and integrating from $y_0 \to y$ and $x_0 \to x$:

$$\frac{dy}{y^3} = -\frac{dx}{x^3} \Rightarrow \frac{1}{y^2} + \frac{1}{x^2} = \frac{1}{y_0^2} + \frac{1}{x_0^2} \qquad (8)$$

This is an equation for a hyperbola shown in figure.



The solution (8) visualized with $x_0 \to \infty$ and $y_0 = 2$