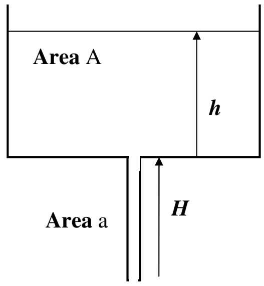
1. **Discharge from a container with a drain pipe**
Water flows out of the reservoir down a pipe of cross-sectional area a, see the figure below. What is the speed of the issuing jet of water? Estimate the time to empty the reservoir. Describe what happens if there is a small hole in the pipe half way down.



Solution: First of all assume that there is no viscosity between fluid and the walls in our construction, then Bernoulli equation is satisfied well between the upper surface and the outflow point of the pipe:

$$\frac{1}{2}V^2 + g(H+h) + \frac{p_0}{\rho} = \frac{1}{2}v^2 + \frac{p_0}{\rho}.$$

From this and the continuity equation of uncompressible fluid

$$VA = va$$

velocities V and v are solved (by lecture notes)[1]:

$$V = a\sqrt{\frac{2g(H+h)}{A^2-a^2}} \qquad v = A\sqrt{\frac{2g(H+h)}{A^2-a^2}}.$$

We denote the height of the fluid surface from the level of out point of the pipe as $\widetilde{h}$. Velocity V gives its rate of change, thus we have differential equation

$$\frac{\mathrm{d}\widetilde{h}}{\mathrm{d}t} = -V = -a\sqrt{\widetilde{h}}\sqrt{\frac{2g}{A^2-a^2}}$$

which has auxiliar form

$$\widetilde{h}^{-\frac{1}{2}}\mathrm{d}\widetilde{h} = -a\sqrt{\frac{2g}{A^2-a^2}}\mathrm{d}t$$

having solution with boundary condition $\widetilde{h}(0) = H + h$ as

$$\widetilde{h}(t) = \left(\sqrt{H+h} - a\sqrt{\frac{g}{2(A^2-a^2)}}t\right)^2. \tag{1}$$

To solve the time to empty the reservoir we have to solve the equation $\widetilde{h}(T) = H$ meaning in terms of equation (1) that

$$\sqrt{H} = \sqrt{H+h} - a\sqrt{\frac{g}{2(A^2-a^2)}}T$$

[1]One can approximate, $A \gg a \to A^2 - a^2 \approx A^2$ but this does not make the solution procedure either easier or more complicated.

which has solution

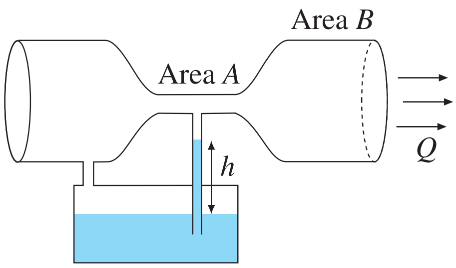
$$T = \frac{\sqrt{2(A^2 - a^2)}}{a\sqrt{g}} \left(\sqrt{h+H} - \sqrt{H}\right).$$

Fluid is incompressible, $va = v_{\text{half point}}a$, meaning that the velocity of the fluid is the same in all points in the pipe. Particularly, at the point $y = H/2$ velocity is v, thus from Bernoulli equation between points $y = 0$ and $y = H/2$ we get:

$$\frac{1}{2}v^2 + g\frac{H}{2} + \frac{p_1}{\rho} = \frac{1}{2}v^2 + \frac{p_0}{\rho} \qquad \Rightarrow \qquad p_1 = p_0 - \rho g\frac{H}{2}.$$

Water is not coming out from small hole. Air is trying to compensate the negative pressure difference by flowing into the pipe from the small aperture.

2. **Venturi flow**
Air is drawn at volume rate Q along a horizontal pipe through a contraction. The pipe is connected to a water tank as sketched in the figure below. Estimate the height h for which water can be sucked into the vertical pipe attached at the constriction.



Solution: The small pipe connecting lower water reservoir and the contraction is so small that it is resonable to assume its effect to the flow neglible. This assumtion includes the fact that the horizontal water flow is not going into the small connection pipe. From the conservation of the volume (flow), it is deduced that

$$Q = v_b B = v_a A,$$

where v_b and v_a are velocities of flow in parts of the construction at area B and area A, respectively.
In addition to that, from Bernoulli equation (ignoring gravity in the horizontal flow), the pressure difference can be solved in those parts:

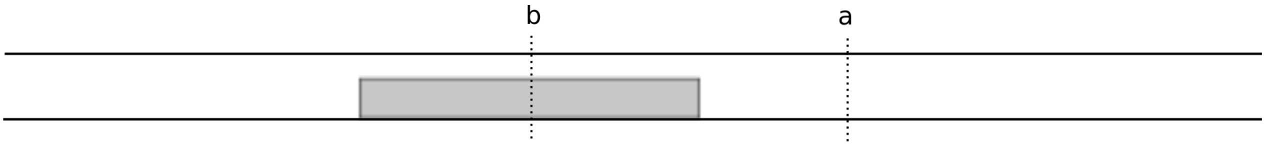
$$\frac{1}{2}v_b^2 + \frac{p_b}{\rho_A} = \frac{1}{2}v_a^2 + \frac{p_a}{\rho_A} \qquad \Rightarrow p_b - p_a = \rho_A \frac{v_a^2 - v_b^2}{2} \qquad \Rightarrow p_b - p_a = \frac{\rho_A Q^2}{2}\frac{B^2 - A^2}{A^2 B^2}.$$

This pressure difference is compensated by the sucked water pilar of height h:

$$gh\rho_W = p_b - p_a \qquad \Rightarrow h = \frac{Q^2 \rho_A}{2g\rho_W}\frac{B^2 - A^2}{A^2 B^2}.$$

Notice the fraction of the air and water densities in the above expression.

3. **Train in a tunnel**
A train travels at speed 150 km/h in a tunnel. How is the air pressure inside the train modified compared to the case that the train would be stationary. Assume the dimensions of the train are width 3 m, height 4 m and length 100 m and the corresponding dimensions of the tunnel are 5 m, 7m and 2 km. (Hint: do all possible simplifying assumptions so that you still get a non-vanishing result.)



Solution:
Let's first notice that the Reynolds number is large $\text{Re} = Va/\nu = 2.77 \cdot 10^7$, where the velocity of the train is $V = 41.67$ m/s, kinematic viscosity of air is $\nu = 1.5 \cdot 10^{-5}$ m^2/s and the characteristic length scale $a = 1$ m. This implies that the thickness of the separation layer around the train is of the order of $d = L/\sqrt{\text{Re}} \approx 0.02$ m, where $L = 100$ m denotes the length of the train. Thus, it is reasonable to ignore viscosity and effects of the wake and the separation layer.
Method 1 for the velocity of air v: Let us consider the situation in the rest frame of the train, where the mass flow of the air at the locations (a) and (b) is

$$Q_a = A_a V \rho \qquad\qquad Q_b = A_b v' \rho.$$

Above, the cross sectional areas at the locations of (a) and (b), respectively, A_a and $A_b = A_a - A_t$, are expressed with the cross sectional areas of the tunnel $A_a = hw$ and the train $A_t = h_t w_t$. Brutally assuming that the air is incompressible, the mass flows are equal and the average velocity of the air around the train is

$$v' = \frac{A_a}{A_a - A_t} V.$$

In the rest frame of the tunnel, the velocity of the air around the train is

$$v = v - V = \frac{A_t}{A_a - A_t} = 21.73 \text{ m/s}.$$

Method 2 for the velocity of air v: This consideration is done in the rest frame of the tunnel. Let us assume that the train does not push air in front of it. Meaning that the air displaced in front oft the moving train flows to the back of the train in the volume between the tunnel and the train. Based on this and the assumption of incompressible air, we formulate using the conservation of mass flow that

$$\rho A_t V = \rho (A_a - A_t) v \quad \Rightarrow \quad v = \frac{A_t}{A_a - A_t} V,$$

giving the same result as above.

Now, in the rest frame of the tunnel, we may apply the Bernoulli equation with respect to the points (a) and (b). Now, at (a) $V_a = 0$ and $p_a = p_0$:

$$\frac{1}{2}V_a^2 + \frac{p_0}{\rho} = \frac{1}{2}v^2 + \frac{p_b}{\rho}$$

and solve the pressure difference

$$p_b - p_0 = -\frac{\rho}{2}v^2 = -\frac{\rho_A A_t^2 V^2}{2(A_a - A_t)^2} = -283 \text{ Pa}.$$

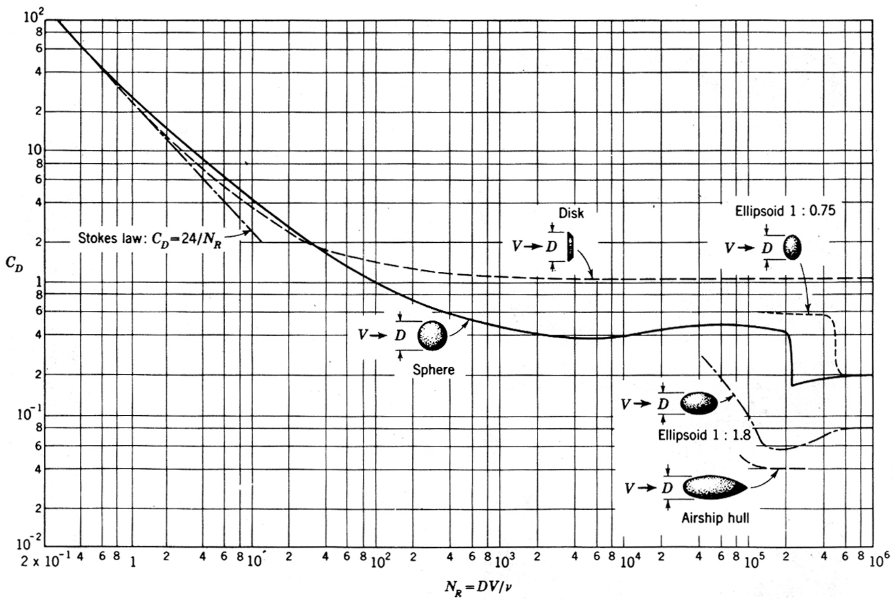
The drawdown of the pressure inside of the train is an noticeable phenomena because it happens rapidly when the train goes into the tunnel. According to the Finnish Meteorological Institute (www.fmi.fi), the air pressure drops 100 Pa in a 8 m change of elevation. Thus, 283 Pa drawdown of the air pressure corresponds elevation change of roughly 22.6 m.

4. **Free fall**

A falling object in a medium reaches a terminal velocity where the gravitational force and the drag force of the medium balance each other. Using the attached graph, estimate the terminal velocities of the following objects in air:

a) a spherical water drop of diameter of 1 mm,
b) a spherical hail of diameter 1 cm,
c) a paratrooper with a parachute diameter 11 m and total mass 160 kg,
d) what would be the velocity of the hail (diameter 1 cm) if the Stokes law were valid?

The drag force has the form $F = \frac{1}{2}C_D \rho A V^2$, and the graph gives the coefficient C_D as a function of the Reynolds number $N_R = DV/\nu$. Here A is the cross-sectional area of the object, V its velocity, ρ is the density of the medium, and ν the kinematic viscosity of the medium. (Hint: Since you do not know the Reynolds number in the beginning, make first a simple guess of C_D, and then correct that once you have an estimate of N_R.



Solution: When a falling object has reached the terminal velocity v_t, the gravitational and drag forces are in balance $F_g = F_d$. With $F_g = \rho_o V_o g$ and the drag force $F_d = \frac{1}{2} C_D \rho A v_t^2$, where ρ_o and V_o denote the density and the volume of the falling object and, we get that

$$v_t = \sqrt{\frac{2\rho_o V_o g}{C_D \rho A}}. \tag{2}$$

Notice, that it is used different notation for the terminal velocity to avoid confusion with symbol for the volume.
The idea is to first make an educated guess of the Reynolds number and based on that look the coefficient C_D from the table and calculate the terminal velocity v_t'. By using it, one can make a better estimate for the Reynolds number and coefficient C_D and get better estimate for v_t.

a) Now with $d = 1 \cdot 10^{-3}$ and $\nu = 1.5 \cdot 10^{-6}$ m^2/s, the Reynolds number

$$N_R = v_t \frac{D}{\nu} = v_t \cdot 66.66 \text{ s/m}.$$

Thus, based on a guesstimate $N_R \approx 10^2$ and $A = \pi d^2/4$ and $V_o = \pi d^3/6$, the first estimate is $C_d = 1.0$ and $v_t' = 3.3$ m/s.
Using $v_t' = 3.3$ m/s, we get that $N_R = 220$ and $C_D = 0.7$ and finally $v_t \approx 4$ m/s.

b) Here, one needs to know density of ice $\rho_I = 917$ kg/m^3. With the exactly same procedure as above, the estimate for the terminal velocity of the hail is $v_t = 16$ m/s. With this velocity, the drag coefficient is $C_D = 0.4$.

c) Here, one can assume that the parachute is disc-shaped. Noticing that

$$N_R = v_t \frac{d}{\nu} = v_t \cdot 7.3 \cdot 10^5 \text{ s/m}$$

and that the value the coefficient $C_D = 1.0$ has no dependence on N_R with such a large Reynolds numbers, there is no need for the iteration loop applied above. Simply,

$$v_t = \sqrt{\frac{2mg}{C_D \rho A}} = 2\sqrt{2}\sqrt{\frac{mg}{C_D \rho \pi d^2}} \approx 5 \text{ m/s}.$$

d) The Stokes law says that the drag coefficient is inversely proportional to the velocity:

$$C_D = \frac{24}{N_R} = \frac{24\nu}{D v_t}.$$

Using this result in Eq. (2) or in $F_g = F_d$, the terminal velocity can be solved straightforwardly in case of a spherical falling hail:

$$v_t = \frac{g}{18} \frac{f_I}{f_A} \frac{d^2}{\nu} = 2776 \text{ m/s} \approx 2800 \text{ m/s}$$

Eight times the speed of sound! Must be unphysical.