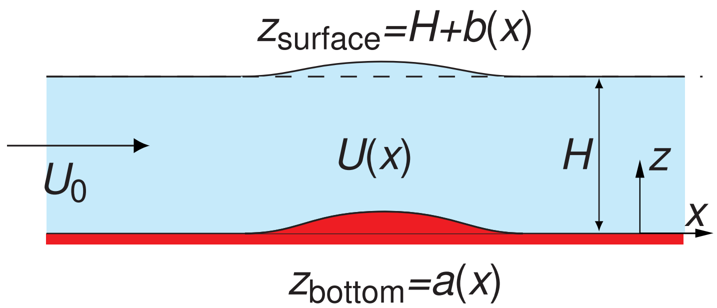
1. **Channel flow**
   Consider water flow in a channel, where the bottom has a smooth hump $z = a(x)$. Using mass conservation and Bernoulli equation (simplest at the surface), calculate the rise $b(x)$ of the free surface $z = H + b(x)$ of the water. Assuming both a and b much smaller than H, solve the coefficient c in the linear relation $b(x) = ca(x)$. Is c always positive?



Solution: The Bernoulli equation along the free surface of the water is

$$\frac{1}{2}U_0^2 + gH + \frac{p_0}{\rho} = \frac{1}{2}U(x)^2 + g[H + b(x)] + \frac{p_0}{\rho} \Rightarrow \quad U(x)^2 = U_0^2 - 2gb(x). \tag{1}$$

The conservation of mass gives

$$U_0 H = U(x)[H - a(x) + b(x)]. \tag{2}$$

We solve $U(x)$ from equation (2)

$$U(x) = U_0 \left(1 + \frac{b-a}{H}\right)^{-1}$$

Squaring this and plugging into Eq. (1) gives

$$U_0^2 \left(1 - \left(1 + \frac{b-a}{H}\right)^{-2}\right) = 2gb.$$

Now, we expand $(1 + (b-a)/H)^{-2}$ to first order in b/H and a/H by exploiting the formula $(1 + x)^\alpha = 1 + \alpha x + \ldots$ for small x:

$$2U_0^2 \frac{b-a}{H} = 2gb \Rightarrow \qquad b\left(1 - \frac{gH}{U_0^2}\right) = a \Rightarrow \qquad b(x) = \frac{a(x)}{1 - gH/U_0^2}.$$

Is c always positive? No, since c is negative, if $gH > U_0^2$, i.e. the if the height of water is large enough with respect to the fluid velocity: $H > U_0^2/g$; or if the fluid velocity is small enough: $U_0 < \sqrt{gH}$. Consider, for example a river with $U_0 = 1$ m/s, $H = 2$ m; thus, using $g = 9.81$ m/s^2, the factor $c = (1 - gH/U_0^2)^{-1} \approx -0.05$, so that an elevation of 20 cm on the bottom of the river would cause a decrease in the surface of the order 1 cm. Notice that the linear $[b(x) = ca(x)]$ approximation breaks down at values $U_0^2 \approx gH$. Notice that the function $a(x)$ can also model a small pit of the channel bottom.

2. **Complex potential**

Show that $\phi = A(x^2 - y^2)$ satisfies $\nabla^2\phi = 0$ and that $\psi = 2Axy$ gives the same velocity field. Show that ϕ and ψ in this case are real and imaginary parts of the complex function $A(x + iy)^2$.

Solution: Now $\nabla^2\phi = (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\phi = A(2 - 2) = 0$. The velocity is obtained from the velocity potential ϕ by taking the gradient,

$$\boldsymbol{v} = \nabla\phi = A(2x\boldsymbol{i} - 2y\boldsymbol{j}).$$

On the other hand, the velocity is found from the vector potential ψ, in two-dimensional case, as

$$\boldsymbol{v} = \boldsymbol{\nabla}\psi \times \boldsymbol{k} = 2A(y\boldsymbol{i} + x\boldsymbol{j}) \times \boldsymbol{k} = 2A(-y\boldsymbol{j} + x\boldsymbol{i}),$$

where we have used $\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{j}$ and $\boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}$. We see that the potentials ϕ and ψ describe the same velocity field. It is quite a trivial to show that ϕ and ψ are the real and imaginary parts of the complex function $A(x + iy)^2$:

$$A(x + iy)^2 = A(x^2 - y^2) + i2Axy = \phi + i\psi.$$

3. **Velocity field in sound wave**

By linearizing the Euler equation and the continuity equation, determine the equation for the velocity field $\boldsymbol{v}'$. Show that this has the plane wave solution

$$\boldsymbol{v}' = A\mathrm{e}^{\mathrm{i}(kx - \omega t)}\boldsymbol{i}$$

and find how the frequency ω depends on the wave vector k.

Solution: We linearize the Euler equation

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} + \rho\boldsymbol{v} \cdot \boldsymbol{\nabla}\boldsymbol{v} = -\boldsymbol{\nabla}p$$

and the continuity equation

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho\boldsymbol{v}) = 0$$

by using $\boldsymbol{v} = \boldsymbol{v}'$, $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$, where p_0 and ρ_0 are constants, and the primed quantities are small. To first order in primed quantities we get

$$\rho_0 \frac{\partial \boldsymbol{v}'}{\partial t} = -\boldsymbol{\nabla} p' = -c^2 \boldsymbol{\nabla} \rho', \tag{3}$$

and

$$\frac{\partial \rho'}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{v}' = 0. \tag{4}$$

In the last equality in eq. (3), we have used $\boldsymbol{\nabla} p' = c^2 \boldsymbol{\nabla} \rho'$, which is valid in case of adiabatic processes ($c^2 = (\partial p / \partial \rho)_s$). Now, taking time derivative of eq. (3) and gradient of eq. (4) gives

$$-\frac{\rho_0}{c^2} \frac{\partial^2 \boldsymbol{v}'}{\partial t^2} = \frac{\partial \boldsymbol{\nabla} \rho'}{\partial t}, \tag{5}$$

and

$$\frac{\partial \boldsymbol{\nabla} \rho'}{\partial t} + \rho_0 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{v}' = 0. \tag{6}$$

Combining these, we find the equation for $\boldsymbol{v}'$:

$$-\frac{\rho_0}{c^2} \frac{\partial^2 \boldsymbol{v}'}{\partial t^2} + \rho_0 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{v}' = 0 \;\Rightarrow\; \frac{\partial^2 \boldsymbol{v}'}{\partial t^2} - c^2 \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{v}' = 0. \tag{7}$$

Inserting $\boldsymbol{v}' = A \mathrm{e}^{\mathrm{i}(kx - \omega t)} \boldsymbol{i}$ gives $-\omega^2 \boldsymbol{v}' + c^2 k^2 \boldsymbol{v}' = 0$, so the dispersion relation becomes $\omega = \pm ck$ and the plane wave is a proper solution.

4. **Attenuation of sound**

Formulate the linearized equations for sound wave including also the dissipative term. Note that you have to use the Navier-Stokes equation for compressible fluid. Form a single equation for $\boldsymbol{v}$. Solve this for a plane wave

$$\boldsymbol{v} = A e^{i(kx - \omega t)} \boldsymbol{i}.$$

Keeping k real, show that ω is complex valued and leads to exponential damping of the amplitude of sound, with damping factor $e^{-\Gamma t}$, $\Gamma = \frac{\omega^2}{2c^2 \rho_0}(K + \frac{4}{3}\mu)$ to first order in viscosity.

(Warning: we have here neglected heat conduction, which leads to additional damping of sound.) Estimate the decay time of sound wave in air of frequency $\omega / 2\pi = 1$ kHz.

Solution: We proceed as in problem 3 before. We use $p = p_0 + p'$ and $\rho = \rho_0 + \rho'$ (we drop the prime from $\boldsymbol{v}$ for shortness). Now the Navier-Stokes equation

$$\rho \frac{\partial \boldsymbol{v}}{\partial t} + \rho \boldsymbol{v} \cdot \boldsymbol{\nabla} \boldsymbol{v} = -\boldsymbol{\nabla} p + \mu \nabla^2 \boldsymbol{v} + (K + \frac{1}{3}\mu) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{v}$$

can be linearized to obtain

$$\rho_0 \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla} p = \mu \nabla^2 \boldsymbol{v} + (K + \frac{1}{3}\mu) \boldsymbol{\nabla} \boldsymbol{\nabla} \cdot \boldsymbol{v}.$$

The time derivative of this is

$$\rho_0 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{\partial \boldsymbol{\nabla} p}{\partial t} = \mu \frac{\partial}{\partial t} \nabla^2 \boldsymbol{v} + (K + \frac{1}{3}\mu) \frac{\partial}{\partial t} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{v}. \tag{8}$$

The gradient of the linearized equation of continuity is (from problem 3)

$$\frac{\partial \boldsymbol{\nabla} p'}{\partial t} + c^2 \rho_0 \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0. \tag{9}$$

Combining equations (8) and (9) gives

$$\rho_0 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} - c^2 \rho_0 \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{v} = \mu \frac{\partial}{\partial t} \nabla^2 \boldsymbol{v} + (K + \frac{1}{3}\mu) \frac{\partial}{\partial t} \boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{v}. \tag{10}$$

For the plane wave

$$\mathbf{v} = A e^{i(kx - \omega t)} \mathbf{i}. \tag{11}$$

we get $\boldsymbol{\nabla}\boldsymbol{\nabla} \cdot \boldsymbol{v} = -k^2 A e^{i(kx-\omega t)} \mathbf{i} = \nabla^2 \boldsymbol{v}$, $\frac{\partial}{\partial t} \nabla^2 \boldsymbol{v} = i\omega k^2 \boldsymbol{v}$, and $\frac{\partial^2}{\partial t^2} \boldsymbol{v} = -\omega^2 \boldsymbol{v}$, so equation (10) takes the form

$$\left[-\rho_0 \omega^2 + \rho_0 c^2 k^2 - i\omega k^2 \left(K + \frac{4}{3}\mu \right) \right] \boldsymbol{v} = 0,$$

which leads to the complex valued dispersion relation

$$\omega^2 = c^2 k^2 - \frac{i\omega k^2}{\rho_0} \left(K + \frac{4}{3}\mu \right), \tag{12}$$

The viscous term is small, and we denote $\omega_0 = ck$. Let us denote $\omega = a + ib$, where a and b are real. Now we have

$$\omega^2 = a^2 + 2iab - b^2 = c^2 k^2 + (-iak^2 + bk^2) \frac{K + \frac{4}{3}\mu}{\rho_0}.$$

The imaginary part of the equation gives

$$b = -\frac{k^2}{2\rho_0} (K + \frac{4}{3}\mu), \quad (a \neq 0),$$

and the real part gives

$$a^2 = c^2 k^2 + bk^2 \frac{K + \frac{4}{3}\mu}{\rho_0} + b^2 = \omega_0^2 - \frac{k^4}{4} \frac{(K + \frac{4}{3}\mu)^2}{\rho_0^2}.$$

Then, application of the formula $(1 + x)^\alpha = 1 + \alpha x + \ldots$ gives

$$a \approx \omega_0 - \frac{1}{2} \frac{k^4 (K + \frac{4}{3}\mu)^2}{4 \omega_0 \rho_0^2},$$

and to first order in viscous terms we have

$$\omega = \omega_0 - i\frac{k^2(K+\frac{4}{3}\mu)}{2\rho_0} = \omega_0 - i\frac{\omega_0^2(K+\frac{4}{3}\mu)}{2c^2\rho_0}.$$

Inserting this into (11) gives

$$\mathbf{v} = Ae^{i(kx-\omega t)}\mathbf{i} = Ae^{-\Gamma t}e^{i(kx-\omega_0 t)},$$

where Γ in the damping factor $e^{-\Gamma t}$ is

$$\Gamma = \frac{\omega_0^2(K+\frac{4}{3}\mu)}{2c^2\rho_0} = \frac{\omega^2(K+\frac{4}{3}\mu)}{2c^2\rho_0}$$

to first order in viscous terms. For a 1 kHz sound wave in air we have $\rho_0 = 1.23$ kg/m^3, $c = 339$ m/s, $\mu = 1.8 \cdot 10^{-5}$ kg/(m s), $K \approx \mu$ and $\omega = 2\pi$ kHz, and $\Gamma \approx 0.00587$ 1/s, or the decay time to e^{th} part is $t = 1/\Gamma \sim 170$ s. In this time, the sound wave travels almost approximately 58 km. It is evident that using the above formulation, the damping coefficient is too low, but Γ although catches important ω^2 dependence, meaning that the low frequency components of sound travels further than the high frequency components, being well known phenomena.