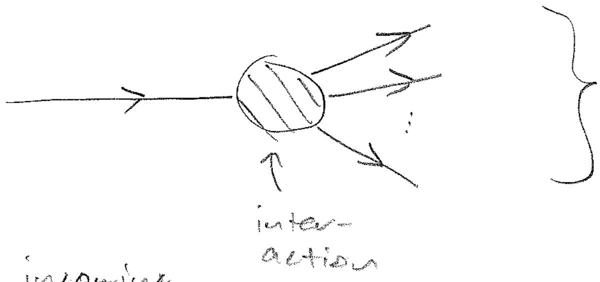
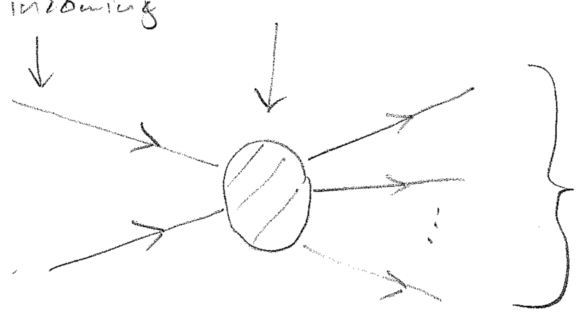
5. Decays

In previous chapters we looked at solutions for free particles. Let us now switch on interactions. This leads to 2 processes:

Decay:



Scattering:
(2-particle)



- Decays:
  - lifetime $\tau = \frac{1}{\Gamma}$, Γ decay rate $\left(N(t) = N_0 e^{-\Gamma t}\right)$
  - decay channel: which particles produced

    e.g. $\tau^- \to \mu^- + \bar{\nu}_\mu + \nu_\tau$

    $\tau^- \to e^- + \bar{\nu}_e + \nu_\tau$

    ....

    * rate for channel i Γ_i; $\Gamma_{tot} = \sum_i \Gamma_i$

    * branching ratio $\equiv \Gamma_i / \Gamma_{tot}$

    = fraction of decays to channel i

How to calculate the rate?

In general, we need to calculate so-called scattering matrix (S-matrix)

$$S_{FI} = \langle \phi_F(t_F) | \hat{U}(t_F, t_I) | \phi_I(t_I) \rangle$$

final state; "time evolution" operator; initial state

take $t_i \to -\infty$; $t_F = +\infty$

Let us see how this happens:

Time evolution - equivalent viewpoints, pictures:

* Schrödinger picture
  - States depend on t, operators not

$$i\partial_t |\psi\rangle_S = \hat{H} |\psi\rangle_S$$

$$\Rightarrow \quad |\psi(t)\rangle_S = e^{-i\hat{H}t} |\psi(0)\rangle_S$$

  - $\langle \hat{A}_S \rangle = {}_S\langle \psi(t) | \hat{A}_S | \psi(t) \rangle_S$

* Heisenberg picture
  - States independent of t, operators depend on t

$$\hat{A}_H \equiv e^{i\hat{H}t} \hat{A}_S e^{-i\hat{H}t}$$

$$\Rightarrow \quad i\frac{\partial}{\partial t} \hat{A}_H = [\hat{A}_H, \hat{H}]$$

  - $\langle \hat{A}_H(t) \rangle = {}_S\langle \psi(0) | \hat{A}_H(t) | \psi(0) \rangle_S = \langle \hat{A}_S \rangle$

* Interaction picture (Dirac-picture)

- Assume $\underline{\hat{H} = \hat{H}_0 + g\hat{V}}$, g small

$\hat{H}_0$: free Hamiltonian, $\hat{V}$: interaction

- Define

$$|\psi(t)\rangle_I = e^{i\hat{H}_0 t}|\psi(t)\rangle_S = e^{i\hat{H}_0 t} e^{-i\hat{H}t}|\psi(0)\rangle_S$$

$$\hat{A}_I(t) = e^{i\hat{H}_0 t}\hat{A}_S e^{-i\hat{H}_0 t}$$

$$\hat{V}_I(t) = e^{i\hat{H}_0 t}\hat{V} e^{-i\hat{H}_0 t}$$

$$\Rightarrow \quad i\frac{d}{dt}|\psi(t)\rangle_I = e^{i\hat{H}_0 t}(-\hat{H}_0 + \hat{H})|\psi(t)\rangle_I$$

$$= g\hat{V}_I(t)|\psi(t)\rangle_I$$

$$i\frac{d}{dt}\hat{A}_I(t) = [\hat{A}_I(t), \hat{H}_0]$$

(if $g\hat{V} = 0$, equivalent to Heisenberg picture)

- Define time evolution operator

$$|\psi(t)\rangle_I = \hat{U}_I(t, t_0)|\psi(t_0)\rangle_I$$

$$\Rightarrow \quad i\frac{d}{dt}\hat{U}_I(t, t_0) = g\hat{V}_I(t)\hat{U}_I(t, t_0) \qquad (*)$$

Note: $\hat{U}_I(t_0, t_0) = \mathbb{1}$

* All pictures equivalent - choose suitable one for computations

* Interaction picture $\longleftrightarrow$ perturbation theory

Clearly, solution for (*) is

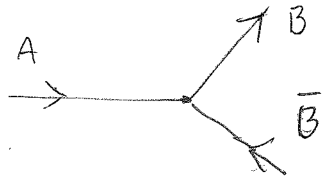
$$\hat{U}_I(t,t_0) = \mathbb{1} - ig\int_{t_0}^{t} dt' \; \hat{V}_I(t')\,\hat{U}_I(t',t_0)$$

This can be iterated to give perturbation series:

$$\hat{U}_I(t,t_0) = \mathbb{1} - ig\int_{t_0}^{t} \hat{V}_I(t')\,dt' + \mathcal{O}(g^2)$$

(Exercise: do this to order g^2)

Let us apply this to decay process.
For concreteness, we take process where heavy particle A decays into light particle + antiparticle B, $\bar{B}$



(This could be an effective theory for decay of ρ-meson into π^+, π^- - in QCD the process is much more complicated.)

Interaction:

$$g\hat{V}_I = \int d^3\bar{x}\; \underbrace{M}_{\text{constant (here)}}\; \hat{\phi}_B(\bar{x},t)\,\hat{\phi}^{+}_{B}(\bar{x},t)\,\hat{\phi}_A(\bar{x},t)$$

Here

destroys incoming A

$$\hat{\phi}_A(\bar{x},t) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[\hat{a}^{(A)}_{\bar{p}} e^{-ip\cdot x} + \hat{a}^{\dagger(A)}_{\bar{p}} e^{ip\cdot x} \right] \in \mathbb{R}$$

creates outgoing $\bar{B}$

$$\hat{\phi}_B(\bar{x},t) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[\hat{a}^{(B)}_{\bar{p}} e^{-ip\cdot x} + \hat{b}^{\dagger(B)}_{\bar{p}} e^{ip\cdot x} \right]$$

$$\hat{\phi}^{\dagger}_B(\bar{x},t) = \int_{d^3\bar{p}} \cdots \left[\hat{a}^{\dagger(B)}_{\bar{p}} e^{ip\cdot x} + \hat{b}^{(B)}_{\bar{p}} e^{-ip\cdot x} \right]$$ creates outgoing B

Thus, the non-trivial part of <u>S-matrix</u> in <u>1st order</u> is

$$S_{FI} = -i \langle B(\bar{p}_1)\,\bar{B}(\bar{p}_2) | \int dt\, d^3\bar{x}\, M\, \hat{\phi}^{\dagger}_B \hat{\phi}_B \hat{\phi}_A | A(\bar{q}) \rangle$$

$$= -i \int dt \int d^3\bar{x}\, M \frac{1}{\sqrt{(2\pi)^3 2E_{\bar{q}}}\sqrt{(2\pi)^3 2E_{\bar{p}_1}}\sqrt{(2\pi)^3 2E_{\bar{p}_2}}} e^{i(p_1+p_2-q)\cdot x}$$

$$= -i(2\pi)^4 \delta^{(4)}(p_1+p_2-q) \underbrace{\frac{M}{\sqrt{(2\pi)^3 2E_{\bar{q}}}\sqrt{(2\pi)^3 2E_{\bar{p}_1}}\sqrt{(2\pi)^3 2E_{\bar{p}_2}}}}_{\text{Transfer matrix element } T_{FI}}$$

Amplitude

4-momentum conservation comes from here (energy + momentum)

Transfer matrix element T_{FI}

(Here $|A(\bar{q})\rangle = \hat{a}^{\dagger(A)}_{\bar{q}}|0\rangle$; $\langle B(\bar{p}_1)\bar{B}(\bar{p}_2)| = \langle 0| \hat{a}^{(B)}_{\bar{p}_1} \hat{b}^{(B)}_{\bar{p}_2} = \left(\hat{a}^{(B)\dagger}_{\bar{p}_1} \hat{b}^{(B)\dagger}_{\bar{p}_2}|0\rangle\right)^{\dagger}$)

From the S-matrix we get the probability by squaring: $p \sim |\langle F|U|I\rangle|^2 \sim |S_{FI}|^2$

$$|S_{FI}|^2 = (2\pi)^4 \delta^{(4)}(p_1+p_2-q)\,(2\pi)^4 \delta^{(4)}(0) \frac{|M|^2}{(2\pi)^3 2E_{\bar{p}_1} (2\pi)^3 2E_{\bar{p}_2} (2\pi)^3 2E_{\bar{q}}}$$

Double δ-function: assume again finite V and time T; then

$$\delta^{(4)}(0) \hat{=} \frac{V \cdot T}{(2\pi)^4} \leftarrow \text{4-volume}$$

⇒ we obtain decay rate $\Gamma = \frac{|S_{FI}|^2}{T}$ by dividing out T.

- Above was actually $\Gamma(\bar{q} \to \bar{p}_1, \bar{p}_2)$. To get total rate, we should integrate over possible final state $\bar{p}$'s:

$$\Gamma = \int d^3\bar{p}_1\, d^3\bar{p}_2\, (2\pi)^4 \delta^{(4)}(p_1+p_2-q) \frac{V}{(2\pi)^3} \frac{|M|^2}{2E_{\bar{q}} (2\pi)^3 2E_{\bar{p}_1} (2\pi)^3 2E_{\bar{p}_2}}$$

- Still one problem: incoming state was normalised to infinity; $\langle A(\bar{q})|A(\bar{q})\rangle = \delta^{(3)}(0) \hat{=} \frac{V}{(2\pi)^3}$.

To normalise to single particle, we demand $\langle A|A\rangle = 1$, and divide by $V/(2\pi)^3$

Thus, we obtain

$$\boxed{\Gamma_{A\to B\bar{B}} = \frac{1}{2E_{\bar{q}}}\int\frac{d^3\bar{p}_1}{(2\pi)^3 2E_{\bar{p}_1}}\int\frac{d^3\bar{p}_2}{(2\pi)^3 2E_{\bar{p}_2}}(2\pi)^4\delta^{(4)}(p_1+p_2-q)\cdot|M|^2}$$

Fermi's golden rule for decays

For n-particle decays

$$\Gamma_{A\to 1,2,3..n} = \frac{1}{2E_{\bar{q}}}\cdot \underset{\uparrow}{C}\cdot \underbrace{\int\left\{\prod_{i=1}^{n}\frac{d^3\bar{p}_i}{(2\pi)^3 2E^{(i)}_{\bar{p}_i}}\right\}(2\pi)^4\delta^{(4)}\left(\sum_i p_i - q\right)}_{\text{Phase space integration}}|M|^2$$

Statistical factor: if j identical particles are produced, $\underline{C = 1/j!}$

In general, M can have $\bar{p}_i, \bar{q}$-dependence (not in our example)

These results are valid for general 2-body (n-body) decays, not only to $A\to B\bar{B}$. (subs. $E_{\bar{p}_i}\to E^{(i)}_{\bar{p}_i} = \sqrt{m_i^2+\bar{p}_i^2}$)

The form above is the most general one. Depending on amplitude M, $\bar{p}_1, \bar{p}_2$-integrations can be (at least partially) performed.

- Phase space integration example: decay $A \to 1+2$, and assume M constant: Choose <u>rest frame of A; $\bar{q}=0$</u>

$$\Gamma_{A\to 1,2} = \frac{|M|^2}{2m_A}\int\frac{d^3\bar{p}_1}{(2\pi)^3 2E^{(1)}_{\bar{p}_1}}\int\frac{d^3\bar{p}_2}{(2\pi)^3 2E^{(2)}_{\bar{p}_2}}(2\pi)^4\,\delta(E_{\bar{p}_1}+E_{\bar{p}_2}-m_A)\,\delta^{(3)}(\bar{p}_1+\bar{p}_2)$$

$$= \frac{|M|^2}{2m_A}\int\frac{d^3\bar{p}_1}{(2\pi)^2\,4}\;\frac{\delta\left(m_A-\sqrt{m_1^2+\bar{p}_1^2}-\sqrt{m_2^2+\bar{p}_1^2}\right)}{\sqrt{m_1^2+\bar{p}_1^2}\;\sqrt{m_2^2+\bar{p}_1^2}}$$

Now $\int d^3\bar{p}_1 \to 4\pi\int_0^\infty d\rho\,\rho^2$

Let now $E = \sqrt{m_1^2+\rho^2}+\sqrt{m_2^2+\rho^2}$

$$\Rightarrow dE = \left[\frac{1}{\sqrt{m_1^2+\rho^2}}+\frac{1}{\sqrt{m_2^2+\rho^2}}\right]\rho\,d\rho$$

$$= \frac{E}{\sqrt{m_1^2+\rho^2}\sqrt{m_2^2+\rho^2}}\,\rho\,d\rho$$

$$\Rightarrow \Gamma_{A\to 1,2} = \frac{|M|^2}{8\pi\,m_A}\int_{m_1+m_2}^{\infty} dE\;\rho(E)\cdot\frac{1}{E}\,\delta(m_A-E)$$

$$= \underline{\frac{|M|^2}{8\pi m_A^2}\,\rho(m_A)}\;, \quad m_1+m_2<m_A$$

where $\rho(m_A)$ can be solved from

$$m_A^2 = m_1^2+m_2^2+2\rho^2+2\sqrt{(m_1^2+\rho^2)(m_2^2+\rho^2)}$$

$$\Rightarrow (m_A^2-m_1^2-m_2^2-2\rho^2)^2 = 4\left(m_1^2m_2^2+(m_1^2+m_2^2)\rho^2+\rho^4\right)$$

$$\Rightarrow m_A^4+m_1^4+m_2^4+\cancel{4\rho^4}-2m_A^2(m_1^2+m_2^2)+2m_1^2m_2^2-4\rho^2(m_A^2-\cancel{m_1^2}-\cancel{m_2^2})$$
$$= 4m_1^2m_2^2+\cancel{4\rho^2(m_1^2+m_2^2)}+\cancel{4\rho^4}$$

$$\Rightarrow \rho^2 = \frac{1}{4m_A^2}\left(m_A^4+m_1^4+m_2^4-2m_A^2(m_1^2+m_2^2)-2m_1^2m_2^2\right)$$

Thus, in this case final result is

$$\Gamma_{A\to 1,2} = \frac{|M|^2}{16\pi\, m_A^3}\left(m_A^4 + m_1^4 + m_2^4 - 2m_A^2(m_1^2+m_2^2) - 2m_1^2 m_2^2\right)^{1/2}$$

If $m_1 = m_2 = 0$, this simplifies to

$$\Gamma = \frac{|M|^2}{16\pi\, m_A}$$

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6 Scattering

- Elastic scattering: $A + B \to A + B$
- Inelastic scattering: others,

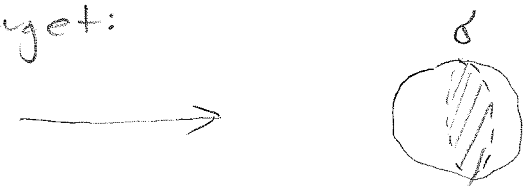
  $A + B \to C + D$

  $A + B \to A + B + C$

  ......

- Cross-section σ

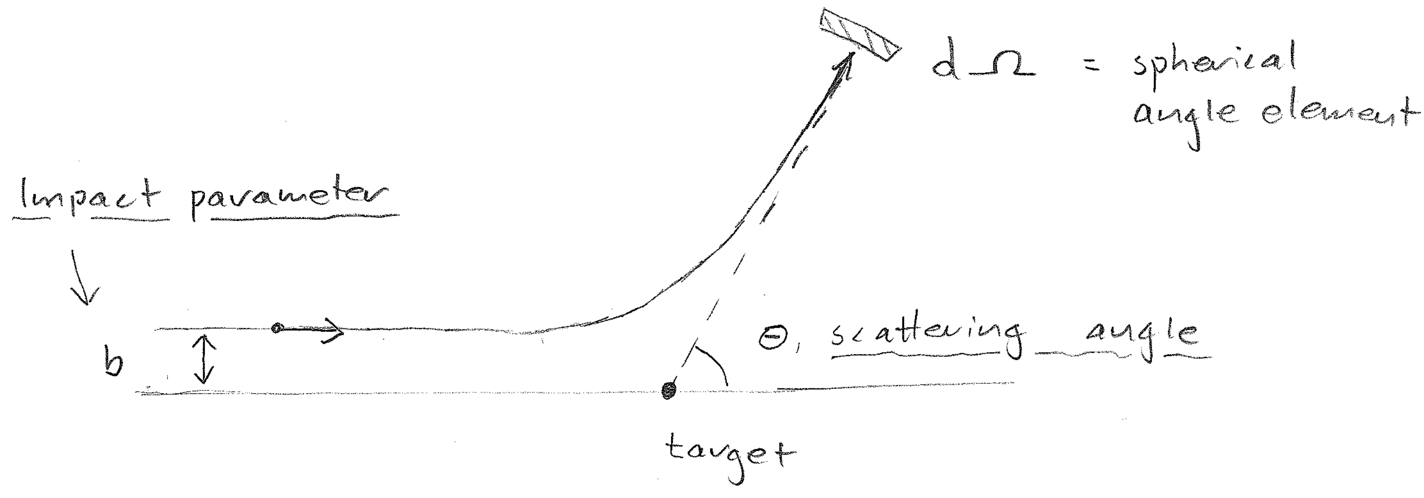
  * Classically, cross-sectional area of the target:



  * Quantum mechanics: σ can depend dramatically on the energy of the collision. At resonance energy looks exceptionally large!

- Exclusive process: all products identified
- Inclusive process: only part identified; e.g.

$$A + B \to C + (\text{anything})$$



$$d\Omega = \sin\theta \, d\theta \, d\phi \quad ; \quad \int d\Omega = 4\pi$$

- Differential cross-section $\frac{d\sigma}{d\Omega}$: scattering cross-section into $d\Omega$, to direction (θ, ϕ)

$$\sigma = \int d\Omega \, \frac{d\sigma}{d\Omega}$$

- Luminosity $L_{in} \equiv \dfrac{d^2 N_{in}}{dt\, dA}$: incoming particle density; particles/time/area unit

Now

$$\frac{d N_{out}}{dt\, d\Omega} = L_{in} \frac{d\sigma}{d\Omega}$$

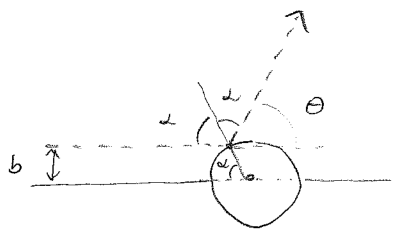
particles/time/space angle to direction θ, ϕ

$$[L_{in}] = \frac{1}{m^2 s} \quad ; \quad [\sigma] = \left[\frac{d\sigma}{d\Omega}\right] = m^2 \sim \text{area}$$

- Common unit for σ: $1 \text{ barn} = (10 \text{ fm})^2$
$= 10^{-24} \text{ cm}^2$

(Name comes from the fact that in particle physics this is very large, easy to hit "barn door"!)

Example: classical hard sphere scattering
(point particle + hard sphere)



$b = R \sin \alpha$

$2\alpha + \theta = \pi$

$\Rightarrow b = R \sin\left(\frac{\pi}{2} - \frac{\theta}{2}\right) = R \cos\frac{\theta}{2}$

Luminosity $L = \dfrac{d^2 N_{in}}{dt\, dA}$

- Polar angle ϕ constant

- How many particles scatter to angle $(\theta, \theta + d\theta)$?

* All which have impact parameter between b, $b + db$, where

$$db = -R \sin\frac{\theta}{2} \frac{d\theta}{2}$$

$$\left.\begin{aligned} d\Omega &= d\phi \sin\theta\, d\theta \\ \text{Now} \quad dA &= |db| \cdot b\, d\phi \end{aligned}\right\} \phi \text{ constant}$$

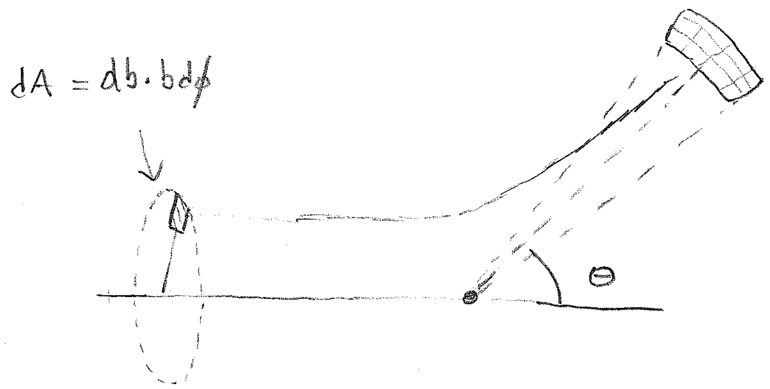
$$= \frac{R \sin\frac{\theta}{2}}{2} b\, d\theta\, d\phi$$

$$= \frac{R^2}{2} \sin\frac{\theta}{2} \cos\frac{\theta}{2}\, d\theta\, d\phi \equiv \frac{R^2}{4} \sin\theta\, d\theta\, d\phi$$

Thus,
$$\underbrace{\frac{d^2N}{dt\,d\Omega}} = \frac{d^2N}{dt\,d\phi\,d\theta\,\sin\theta} = \underbrace{\frac{d^2N}{dt\,dA}}_{L_{in}}\;\frac{R^2/4\,\sin\theta}{\sin\theta}$$
$$= L_{in}\cdot\frac{R^2}{4}$$

$$\Rightarrow\ \frac{d\sigma}{d\Omega} = \frac{R^2}{4} \quad\Rightarrow\quad \sigma = \int d\Omega\,\frac{R^2}{4} = \underline{\pi R^2}$$

The classical x-section! (of course...)



- All particles which go through area dA (in time dt) will exit through spherical angle $d\Omega$. To get $d\sigma/d\Omega$, we need to do change of variables → |jacobian|.

- <u>Note</u>: Luminosity $L_{in} = \dfrac{d^2N_{in}}{dt\,dA}$ = particles going through dA!

$$\Rightarrow L_{in} = \underbrace{\frac{N_{in}}{V}}_{\text{density of incoming particles}} v_{in} = \frac{1}{V}\,v_{in}$$

↖ if we normalise $\langle A|A\rangle = 1$, 1 particle$/V$!