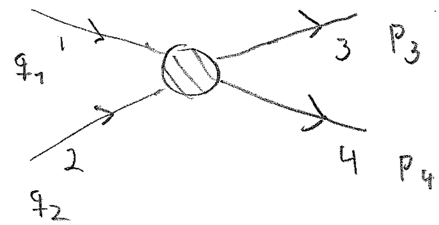
6.2. Golden rule for scattering

Consider process (2-particle scattering)



(for simplicity, assume real...)

Assume $\hat{H} = \hat{H}_0 + g\hat{V}$; $g\hat{V}_I = \int d^3x\, \mathcal{M}\, \hat{\phi}_1 \hat{\phi}_2 \hat{\phi}_3 \hat{\phi}_4$

$$\Rightarrow \quad S_{FI} = -i \langle \phi_3(\bar{p}_3)\, \phi_4(\bar{p}_4) | \int dt\, g\hat{V}_I | \phi_1(\bar{q}_1)\, \phi_2(\bar{q}_2) \rangle$$

As before:

* Incoming particle: $\dfrac{1}{\sqrt{(2\pi)^3\, 2E_{\bar{q}}}}\, e^{-iq\cdot x}$

* Outgoing particle: $\dfrac{1}{\sqrt{(2\pi)^3\, 2E_{\bar{p}}}}\, e^{ip\cdot x}$

$$\Rightarrow S_{FI} = -i(2\pi)^4\, \delta^{(4)}(p_3+p_4-q_1-q_2)\, \frac{\mathcal{M}}{\sqrt{(2\pi)^3 2E_{\bar{p}_3}}\sqrt{(2\pi)^3 2E_{\bar{p}_4}}\sqrt{(2\pi)^3 2E_{\bar{q}_1}}\sqrt{(2\pi)^3 2E_{\bar{q}_2}}}$$

$$\Rightarrow |S_{FI}|^2 = (2\pi)^4\, \delta^{(4)}(p_3+p_4-q_1-q_2)\cdot VT\, \frac{|\mathcal{M}|^2}{\prod_{i=\text{part.}} (2\pi)^3\, 2E_{\bar{p}_i}}$$

* Normalisation: incoming particles were normalised to $\delta^{(3)}(0)$. If we want to look at the probability of scattering of only 2 particles, divide twice by $V/(2\pi)^3$!

* rate $\Gamma = |S|^2/T$

* integrate over outgoing $\bar{p}$:s :

$\Rightarrow$ total scattering rate

$$\frac{dN_{out}}{dt} = \frac{1}{V \cdot 2E_{\bar{q}_1} 2E_{\bar{q}_2}} \int \frac{d^3\bar{p}_3}{(2\pi)^3 2E_{\bar{p}_3}} \int \frac{d^3\bar{p}_4}{(2\pi)^3 2E_{\bar{p}_4}} \times$$

$$(2\pi)^4 \delta^{(4)}(p_1+p_2-q_1-q_2)\, |M|^2$$

* Here N_{out} is scattered particle 3, say. Incoming flux is normalised so that in V there is one particle of type 1 (or 2). Thus, from page 114, if we choose coordinates where 2 is <u>at rest</u>: $(E_{\bar{q}_2} = m_2)$

$$\text{Luminosity} = L_{in} = \frac{1}{V} |\bar{v}_1|$$

* Now cross-section is obtained from

$$\frac{dN_{out}}{dt} = L_{in}\, \sigma \qquad \text{(pg. 112)}$$

$\Rightarrow$ <u>in this coordinate system</u>

$$\sigma = \frac{dN_{out}}{dt} \Big/ L_{in} = \frac{1}{4|\bar{v}_1| E_{\bar{q}_1} m_2} \int d\Phi_2\, |M|^2$$

where phase space integral is defined

$$\int d\Phi_n = \int \left[\prod_{i=1}^{n} \frac{d^3\bar{p}_i}{(2\pi)^3 2E_{\bar{p}_i}} \right] (2\pi)^4 \delta^{(4)}\left(\sum_{j=1}^{n} p_j - q_1 - q_2\right)$$

* We call $F \equiv 4|\bar{v}_1| E_{q_1} m_2$ flux factor.

With this cross-section is simply

$$\boxed{\sigma = \frac{1}{F} \int d\Phi_2 |M|^2}$$

Golden rule for scattering $(2 \to 2)$

* Differential cross-section $\frac{d\sigma}{d\Omega}$ is obtained by leaving out one angle-integral in $\int d\Phi$.

6.3 Flux factor

- Because scattering is a physical process it cannot depend on Lorentz-frame. Thus, σ must be Lorentz-invariant. Similarly, $\int d\Phi$ and $|M|^2$ are L-invariants $\Rightarrow$ F must be too. Lorentz-invariant form is

$$F = 4\sqrt{(q_1 \cdot q_2)^2 - m_1^2 m_2^2}$$

F depends only on incoming particles

If we set $\underline{q_1 = (E_{\bar{q}_1}, \bar{q}_1)}$; $\underline{q_2 = (m_2, \bar{0})}$

$$(q_1 \cdot q_2)^2 - m_1^2 m_2^2 = E_{\bar{q}_1}^2 m_2^2 - m_1^2 m_2^2 = \bar{q}_1^{\,2} m_2^2 \Rightarrow$$

$$F = 4 |\bar{q}_1| m_2 = 4 \frac{|\bar{q}_1|}{E_{\bar{q}_1}} E_{\bar{q}_1} m_2 = \underline{4 |\bar{v}_1| E_{\bar{q}_1} m_2} . \quad \square$$

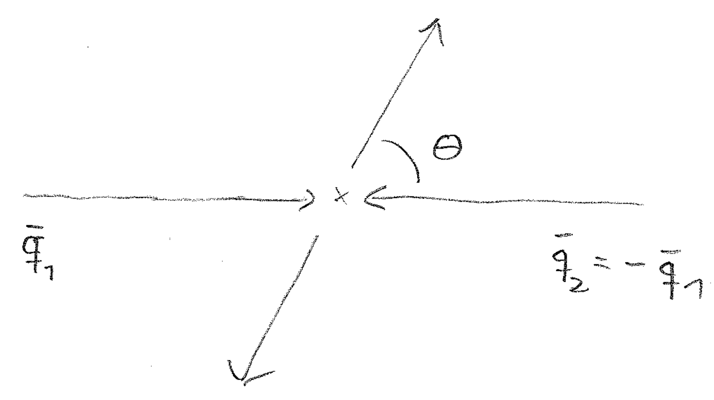
Besides the rest frame of the "target" (2), center-of-mass frame is important. Then

$$q_1 = (E_1, \bar{q}_1) \;; \qquad q_2 = (E_2, -\bar{q}_1)$$

$$(q_1 \cdot q_2)^2 = (E_1 E_2 + \bar{q}_1^{\,2})^2 = (m_1^2 + \bar{q}_1^{\,2})(m_2^2 + \bar{q}_1^{\,2}) + (\bar{q}_1^{\,2})^2 + 2 E_1 E_2 \bar{q}_1^{\,2}$$

$$\begin{aligned}(q_1 \cdot q_2)^2 - m_1^2 m_2^2 &= (m_1^2 + m_2^2 + 2 E_1 E_2 + 2 \bar{q}_1^{\,2}) \, \bar{q}_1^{\,2} \\ &= (E_1 + E_2)^2 \, \bar{q}_1^{\,2}\end{aligned}$$

$$\Rightarrow \quad \underline{F = 4 (E_1 + E_2) |\bar{q}_1|}$$



F can be re-expressed using important kinematical variable (one of Mandelstam vars)

$$\underline{s \equiv (q_1 + q_2)^2}$$

Dot product $\rightarrow$ <u>Lorentz-invariant</u>

$$s = q_1^2 + q_2^2 + 2 q_1 \cdot q_2 = m_1^2 + m_2^2 + 2 q_1 \cdot q_2$$

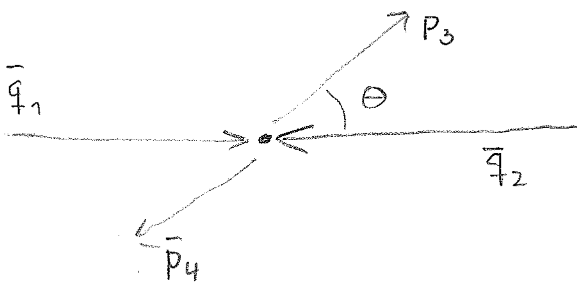
$$\Rightarrow \quad F = 2\sqrt{(s - m_1^2 - m_2^2)^2 - 4 m_1^2 m_2^2}$$

In center-of-mass frame $s = (q_1 + q_2)^2 = (E_1 + E_2)^2$

$\Rightarrow$ $\underline{\sqrt{s} = E_1 + E_2}$ = center-of-mass energy!

6.4 Phase space integration

- Consider $2 \rightarrow 2$ scattering. In <u>c-o-m frame</u>



Total $\sigma = \frac{1}{F} \int d\Phi_2 |M|^2$

Now $F = 4(E_1 + E_2)|\bar{q}_1|$. Thus,

unintegrated

$$d\sigma = \frac{1}{4(E_1+E_2)|\bar{q}_1|} \frac{d^3\bar{p}_3}{(2\pi)^3 2E_3} \frac{d^3\bar{p}_4}{(2\pi)^3 2E_4} (2\pi)^4 \delta^{(4)}(p_3+p_4-q_1-q_2) |M|^2$$

$$= \frac{1}{(8\pi)^2} \frac{|M|^2}{|\bar{q}_1|(E_1+E_2)E_3E_4} d^3\bar{p}_3 \, d^3\bar{p}_4 \, \delta^{(4)}(p_3+p_4-q_1-q_2)$$

• In <u>c-o-m frame</u> $\bar{q}_2 = -\bar{q}_1 \Rightarrow \delta^{(4)} = \delta^{(3)}(\bar{p}_3+\bar{p}_4)\,\delta(E_3+E_4-E_1-E_2)$

• Thus, integral over $\bar{p}_4 \rightarrow \bar{p}_4 = -\bar{p}_3$

$$d\sigma = \frac{1}{(8\pi)^2} \frac{|M|^2}{|\bar{q}_1|(E_1+E_2)} d^3\bar{p}_3 \frac{\delta\left(\sqrt{m_3^2+\bar{p}_3^2}+\sqrt{m_4^2+\bar{p}_3^2}-E_1-E_2\right)}{\sqrt{m_3^2+\bar{p}_3^2}\sqrt{m_4^2+\bar{p}_3^2}}$$

(call this still $d\sigma$, although $d\sigma = \int d^3\bar{p}_4 \, d\sigma_{previous}$)

• Can we integrate this further without knowing $|M|^2$? $|M|^2$ is scalar, with momentum dependence $|M|^2(\bar{q}_1, \bar{q}_2, \bar{p}_3, \bar{p}_4) \Rightarrow |M|^2(\bar{q}_1, -\bar{q}_1, \bar{p}_3, -\bar{p}_3)$

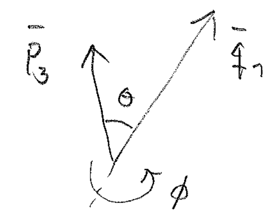
$\Rightarrow |M|^2$ can depend only on $\bar{q}_1^2$, $\bar{p}_3^2$, $\bar{q}_1 \cdot \bar{p}_3$

$= |\bar{q}_1||\bar{p}_3| \cos\theta$

This prevents integration over angle. However, we can use δ-function to do integration over length $|\bar{p}_3|$

Write $d^3\bar{p}_3 = \rho^2 d\rho \, d\Omega$, $d\Omega = \sin\theta \, d\theta \, d\phi$

$\rho = |\bar{p}_3|$



$\Rightarrow$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{1}{|\bar{q}_1|(E_1+E_2)} \int_0^\infty d\rho \, \rho^2 |M|^2 \frac{\delta\left(\sqrt{m_3^2+\rho^2}+\sqrt{m_4^2+\rho^2}-E_1-E_2\right)}{\underbrace{\sqrt{m_3^2+\rho^2}}_{E_3} \; \underbrace{\sqrt{m_4^2+\rho^2}}_{E_4}}$$

- As in page 110, switch integration variable

$$E = E_3 + E_4 = \sqrt{m_3^2+\rho^2} + \sqrt{m_4^2+\rho^2} \qquad (= E_1 + E_2 = \sqrt{s} \text{ in c-o-m!})$$

$$dE = \left(\frac{1}{E_3} + \frac{1}{E_4}\right)\rho \, d\rho = \frac{E}{E_3 E_4} \rho \, d\rho$$

Inverse $\rho(E) = \frac{1}{2E}\sqrt{E^4 + (m_3^2 - m_4^2)^2 - 2E^2(m_3^2+m_4^2)}$ (see 110)

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{1}{|\bar{q}_1|\sqrt{s}} \int_{m_3+m_4}^\infty \frac{dE}{E} \rho(E) |M|^2 \delta(E - \sqrt{s})$$

δ gives $=0$ if $m_3 + m_4 > \sqrt{s} = E_1 + E_2$, i.e. there is not enough energy to create particles 3,4.

Thus,

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \frac{1}{|\bar{q}_1|\sqrt{s}} \frac{\rho(\sqrt{s})}{\sqrt{s}} |M|^2 \, \Theta(\sqrt{s} - m_1 - m_2)$$

function of $|\bar{q}_1|$, $\rho(\sqrt{s})$, $\cos\theta$

* Remember: $\rho(\sqrt{s}) = |\bar{p}_3|$ = momentum required to satisfy the conservation of 4-momentum $p_3 + p_4 = q_1 + q_2$!

* Thus, we can rewrite $d\sigma/d\Omega$ in "familiar" units as (center of mass frame!)

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\,\frac{|\bar{p}_3|}{|\bar{q}_1|}\,\frac{1}{(E_1+E_2)^2}\cdot|M|^2}$$

here $\bar{q}_2 = -\bar{q}_1$; $\bar{p}_4 = -\bar{p}_3$; $E_1 + E_2 = E_{TOT}$ total energy in c-o-m frame. $|M|^2 = |M|^2(|\bar{q}_1|, |\bar{p}_3|, \cos\theta)$

* It is notable that the phase space integration could be performed in $2\to 2$ scattering without information about $|M|$. This is <u>not</u> so in $2\to N$ -scattering!

* Often it is said: M contains the "physics", phase space integration = "only kinematics"; but of course very important.

* Mandelstam variables:

$2 \to 2$ scattering kinematics is often practical to describe in terms of

$$s = (q_1 + q_2)^2$$

$$t = (q_1 - p_3)^2$$

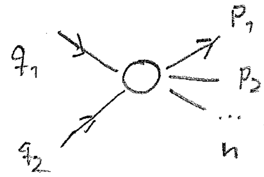
$$u = (q_1 - p_4)^2$$

(only 2 independent.) These are Lorentz-invariant. We had only 2 independent variables, $|\bar{q}_1|$, $\cos\theta$

$\Rightarrow$ s, t sufficient to describe all scattering quantities

* NOTE: if produced particles are identical, we have to multiply above σ by $1/2!$. With j identical particles, by $1/j!$

RECAP: Golden rule for scattering:



$$\sigma = \frac{S}{F} \int d\Phi_n |M|^2$$

$$F = 4\sqrt{(q_1 \cdot q_2)^2 - (m_1 m_2)^2}$$

$$S = \prod_{\substack{j = \text{different} \\ \text{particles}}} \frac{1}{n_j!} \quad ; \quad \sum_j n_j = n$$

$$d\Phi_n = \left[\prod_{i=1}^{n} \frac{d^3 \bar{p}_i}{(2\pi)^3 2E_i} \right] (2\pi)^4 \delta^{(4)}\left(\sum_{i=1}^{n} p_i - q_1 - q_2 \right)$$