8.2 Traces of γ-matrices

* For fermionic cross-sections we repeteadly need to take traces of combinations of γ's.

* Recall:

$$\mathrm{Tr}(A+B) = \mathrm{Tr}\,A + \mathrm{Tr}\,B$$
$$\mathrm{Tr}(\alpha A) = \alpha \,\mathrm{Tr}\,A$$
$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$
$$\mathrm{Tr}\,A = \sum \lambda_i \quad , \quad \lambda \text{ eigenvalue of } A$$

* For 2 complex vectors a, b

$$\underline{a^\dagger b} = a_\alpha^* b_\alpha = b_\alpha a_\alpha^* = \underline{\mathrm{Tr}(b a^\dagger)}$$

* Traces of γ's :

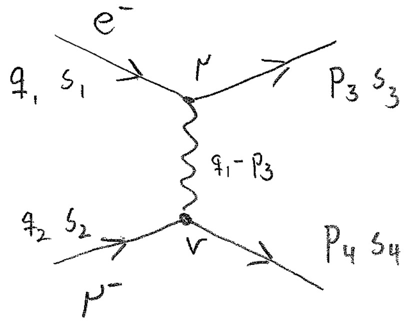
$$\mathrm{Tr}(1) = 4$$
$$\mathrm{Tr}\,\gamma^\mu = 0 \quad \Rightarrow \quad \mathrm{Tr}\,\not{a} = 0$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4 g^{\mu\nu} \qquad \mathrm{Tr}\,\not{a}\not{b} = 4\, a\cdot b$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho = 0$$
$$\mathrm{Tr}\,\underbrace{\gamma^{\mu_1}\cdots\gamma^{\mu_n}}_{\text{odd }\#} = 0 \quad \Rightarrow \quad \mathrm{Tr}\,\underbrace{\not{a}_1\not{a}_2\cdots\not{a}_n}_{\text{odd}} = 0$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$$
$$\Rightarrow$$
$$\mathrm{Tr}\,\not{a}\not{b}\not{c}\not{d} = 4\left(a\cdot b\, c\cdot d - a\cdot c\, b\cdot d + a\cdot d\, b\cdot c\right)$$

* Also :

$$\gamma_\mu\gamma^\mu = 4$$
$$\gamma_\mu\gamma^\nu\gamma^\mu = -2\gamma^\nu \qquad \Rightarrow \gamma_\mu\not{a}\gamma^\mu = -2\not{a}$$
$$\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu} \qquad \Rightarrow \not{a}\not{b} + \not{b}\not{a} = 2\, a\cdot b$$

8.3 $e^- + \mu^-$ -scattering



* To calculate $\mathcal{M}$ we apply the rules in sec. 7.

Follow each fermion line backwards:

$$\mathcal{M} = \underset{\substack{\uparrow \\ \text{outgoing } e^- \\ (124)}}{\bar{u}(p_3, s_3)}\ \underset{\substack{\uparrow \\ \text{vertex} \\ (129)}}{ie\gamma^\mu}\ \underset{\substack{\uparrow \\ \text{incoming } e \\ (124)}}{u(q_1, s_1)} \times \underbrace{\bar{u}(p_4, s_4)\ ie\gamma^\nu\ u(q_2, s_2)}_{\text{myon}}$$

$$\times \underset{\substack{\uparrow \\ \gamma\text{-propagator} \\ (126)}}{\frac{(-i g_{\mu\nu})}{(q_1 - p_3)^2}} \times \underset{\substack{\uparrow \\ \text{overall phase} \\ (127)}}{(i)}$$

$$= \frac{-e^2}{(q_1 - p_3)^2}\ \bar{u}(p_3, s_3)\gamma^\mu u(q_1, s_1) \times \bar{u}(p_4, s_4)\gamma_\mu u(q_2, s_2)$$

This is a number which depends on $\bar{p}$'s and s's. It is easier to evaluate when we calculate $|\mathcal{M}|^2$, especially if we <u>average</u> over s_i's.

* Often (usually) we do not care or know the spin state of the scattered particles, and the incoming particles have arbitrary spin state (with equal probability)

* We take average of the initial spins and sum of the final spins:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{s_1 s_2 s_3 s_4} |M|^2$$

with

$$|M|^2 = \frac{e^4}{(q_1 - p_3)^4} \left[\bar{u}(p_3, s_3)\gamma^\mu u(q_1, s_1)\right]\left[\bar{u}(p_3, s_3)\gamma^\nu u(q_1, s_1)\right]^* \times \left[\bar{u}(p_4, s_4)\gamma_\mu u(q_2, s_2)\right]\left[\bar{u}(p_4, s_4)\gamma_\nu u(q_2, s_2)\right]^*$$

NOTE: each of [ ] is a number, not matrix! Thus it is possible to shuffle those as above.

Now

$$\left[\bar{u}(p_3, s_3)\gamma^\nu u(q_1, s_1)\right]^* = \left[u_3^\dagger \gamma^0 \gamma^\nu u_1\right]^\dagger$$

$$= \left[u_1^\dagger \gamma^{\nu\dagger} \gamma^{0\dagger} u_3\right]$$

$$= \left[u_1^\dagger \gamma^0 \gamma^\nu u_3\right]$$

$$= \bar{u}(q_1, s_1)\gamma^\nu u(p_3, s_3)$$

$$\gamma^{\nu\dagger} = \gamma^0 \gamma^\nu \gamma^0$$

$$\gamma^{0\dagger} = \gamma^0$$

$$\Rightarrow |M|^2 = \frac{e^4}{(q_1-p_3)^4} \, \bar{u}(p_3,s_3)\gamma^\mu u(q_1,s_1)\, \bar{u}(q_1,s_1)\gamma^\nu u(p_3,s_3)$$
$$\times\, \bar{u}(p_4,s_4)\gamma_\mu u(q_2,s_2)\, \bar{u}(q_2,s_2)\gamma_\nu u(p_4,s_4)$$

- Recall completeness relation: (p. 98)

$$\sum_s{}' u(\bar{p},s)\bar{u}(\bar{p},s) = \not{p} + m$$

- Moreover, because $a^\dagger b = \mathrm{Tr}(ba^\dagger) \Rightarrow$

$$\sum_s{}' \bar{u}(\bar{p},s)\, M\, u(\bar{p},s) = \sum_s{}' \mathrm{Tr}\left[M u(\bar{p},s)\bar{u}(p,s)\right]$$
$$= \mathrm{Tr}\left[M(\not{p}+m)\right]$$

any 4x4 matrix

- Thus,

$$\langle |M|^2 \rangle = \frac{e^4}{4(q_1-p_3)^4} \mathrm{Tr}\left[\gamma^\mu(\not{q}_1+m_e)\gamma^\nu(\not{p}_3+m_e)\right] \times$$
$$\mathrm{Tr}\left[\gamma_\mu(\not{q}_2+m_\mu)\gamma_\nu(\not{p}_4+m_\mu)\right]$$

→ Completeness relations got us rid of spinors!

- Use γ-matrix trace results (p. 133)

Now $\underline{\mathrm{Tr}[\gamma^\mu \not{a} \gamma^\nu \not{b}]} = a_\alpha b_\beta \mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta]$

$$= a_\alpha b_\beta \cdot 4(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\nu\alpha}) = \underline{4[a^\mu b^\nu + a^\nu b^\mu - g^{\mu\nu} a\cdot b]}$$

Thus,

$$\mathrm{Tr}[\gamma^\mu(\not{q}_1+m_e)\gamma^\nu(\not{p}_3+m_e)] = 4[q_1^\mu p_3^\nu + q_1^\nu p_3^\mu - g^{\mu\nu}(q_1\cdot p_3 - m_e^2)]$$
$$\mathrm{Tr}[\gamma_\mu(\not{q}_2+m_\mu)\gamma_\nu(\not{p}_4+m_\mu)] = 4[q_{2\mu}p_{4\nu} + q_{2\nu}p_{4\mu} - g_{\mu\nu}(q_2\cdot p_4 - m_\mu^2)]$$

Combining these all we obtain ($g_{\mu\nu} g^{\mu\nu} = g_\mu{}^\mu = 4$)

$$
\begin{aligned}
\langle |M|^2 \rangle = \frac{e^4}{4(q_1 - p_3)^4} \cdot 16 \cdot \Big[& 2\, q_1 \cdot q_2 \; p_3 \cdot p_4 + 2\, q_1 \cdot p_4 \; q_2 \cdot p_3 \\
& - 2\, q_1 \cdot p_3 (q_2 \cdot p_4 - m_\mu^2) - 2\, q_2 \cdot p_4 (q_1 \cdot p_3 - m_e^2) \\
& + \underset{4}{g_\mu{}^\mu} (q_1 \cdot p_3 - m_e^2)(q_2 \cdot p_4 - m_\mu^2) \Big]
\end{aligned}
$$

$$
\begin{aligned}
= \frac{8 e^4}{(q_1 - p_3)^4} \Big[& q_1 \cdot q_2 \; p_3 \cdot p_4 + q_1 \cdot p_4 \; q_2 \cdot p_3 \\
& - q_1 \cdot p_3 \, m_\mu^2 - q_2 \cdot p_4 \, m_e^2 + 2 m_e^2 m_\mu^2 \Big]
\end{aligned}
$$

- This is the final result for $\langle |M|^2 \rangle$ in $e^- + \mu^- \to e^- + \mu^-$ scattering to order e^4.
- Can be inserted in $\frac{d\sigma}{d\Omega}$-formula in p. 122. (or 123)
- Let us look at some limiting cases:

Mott and Rutherford scattering

- Let us take the limit where myon is so heavy ($m_\mu \gg m_e$) that myon recoil can be neglected; thus, μ stays at rest before and after the collision (limit $m_\mu \to \infty$)

In this limit, $q_1 = (E_1, \bar{q}_1)$, $q_2 = p_4 = (m_p, \bar{0})$, $p_3 = (E_1, \bar{p}_3)$; $|\bar{p}_3| = |\bar{q}_1|$

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2} \underbrace{\frac{|\bar{p}_3|}{|\bar{q}_1|}}_{=1} \frac{1}{\underbrace{(E_1+E_2)^2}_{=(m_p + \sqrt{m_e^2 + \bar{q}_1^2})^2}} \langle |M^2| \rangle \simeq \frac{1}{(8\pi)^2} \frac{1}{m_p^2} \langle |M|^2 \rangle$$

$\Rightarrow q_1 \cdot q_2 = E_1 m_p$; $p_3 \cdot p_4 = E_1 m_p$; $q_1 \cdot p_4 = E_1 m_p$

$q_2 \cdot p_3 = E_1 m_p$; $q_1 \cdot p_3 = E_1^2 - \bar{q}_1 \cdot \bar{p}_3 = m_e^2 + |\bar{q}_1|^2 (1 - \cos\theta)$

$q_2 \cdot p_4 = m_p^2$;

$(q_1 - p_3)^2 = 2m_e^2 - 2 q_1 \cdot p_3 = -2|\bar{q}_1|^2 (1 - \cos\theta)$

$$\Rightarrow \langle |M|^2 \rangle = \frac{2e^4}{|\bar{q}_1|^4 (1-\cos\theta)^2} \Big[2 \underbrace{E_1^2}_{=|\bar{q}_1|^2 + m_e^2} m_p^2 - (m_e^2 + |\bar{q}_1|^2 (1-\cos\theta)) m_p^2 + m_e^2 m_p^2 \Big]$$

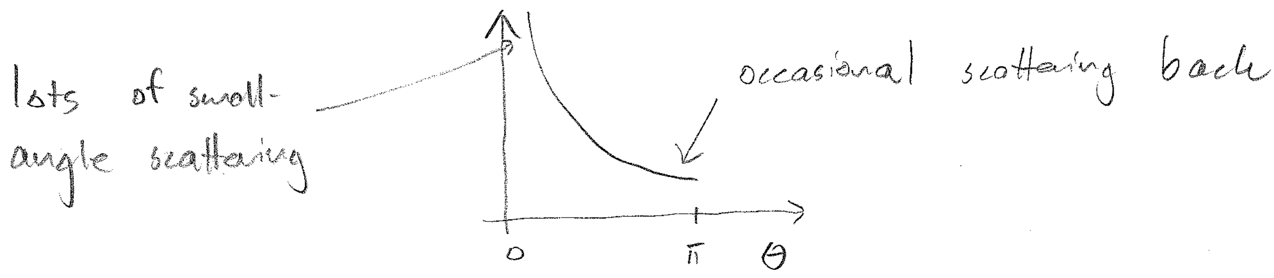
$$= \left[\frac{e^2}{2|\bar{q}_1|^2 \sin^2\frac{\theta}{2}} \right]^2 \Big[2 m_e^2 m_p^2 + |\bar{q}_1|^2 \underbrace{(1 + \cos\theta)}_{2\cos^2\frac{\theta}{2}} m_p^2 \Big]$$

Thus, $$\frac{d\sigma}{d\Omega} \simeq \left[\frac{\alpha}{2|\bar{q}_1|^2 \sin^2\frac{\theta}{2}} \right]^2 \left(m_e^2 + |\bar{q}_1|^2 \cos^2\frac{\theta}{2} \right)$$

Mott formula. No m_p-dependence! It gives also the cross-section for $e^- + p$ -scattering (another $e \to -e$, but in $[\,]^2$ makes no difference)

If e^- is non-relativistic, $|\bar{q}_1| \ll m_e$, and we obtain the Rutherford formula:

$$\frac{d\sigma}{d\Omega} = \left[\frac{\alpha}{2 m_e v^2 \sin^2\frac{\theta}{2}}\right]^2 \qquad (|\bar{q}_1| = m_e|\bar{v}_1|)$$



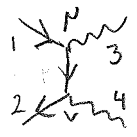
In other processes in pages 130-131 there were 2 diagrams at order e^2. In this case

$$|M|^2 = |M_1 \pm M_2|^2 = |M_1|^2 + |M_2|^2 \pm 2|M_1 M_2|$$

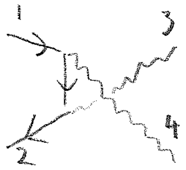
↑ interference!

Relative sign depends on the Feynman rules

EXAMPLE: $e^+ + e^-$ -annihilation



$$M_1 = \bar{v}_2 \, ie\gamma^\nu \epsilon_{4,\nu} \frac{i(\not{p}_1 - \not{p}_3 + m)}{(p_1-p_3)^2 - m^2} ie\gamma^\mu \epsilon_{3,\mu} u_1$$



$$M_2 = \bar{v}_2 \, ie\gamma^\nu \epsilon_{3,\nu} \frac{i(\not{p}_1 - \not{p}_4 + m)}{(p_1-p_4)^2 - m^2} ie\gamma^\mu \epsilon_{4,\mu} u_1$$

To evaluate photon polarisations and calculate σ is rather tedious (242-245 in Griffiths)