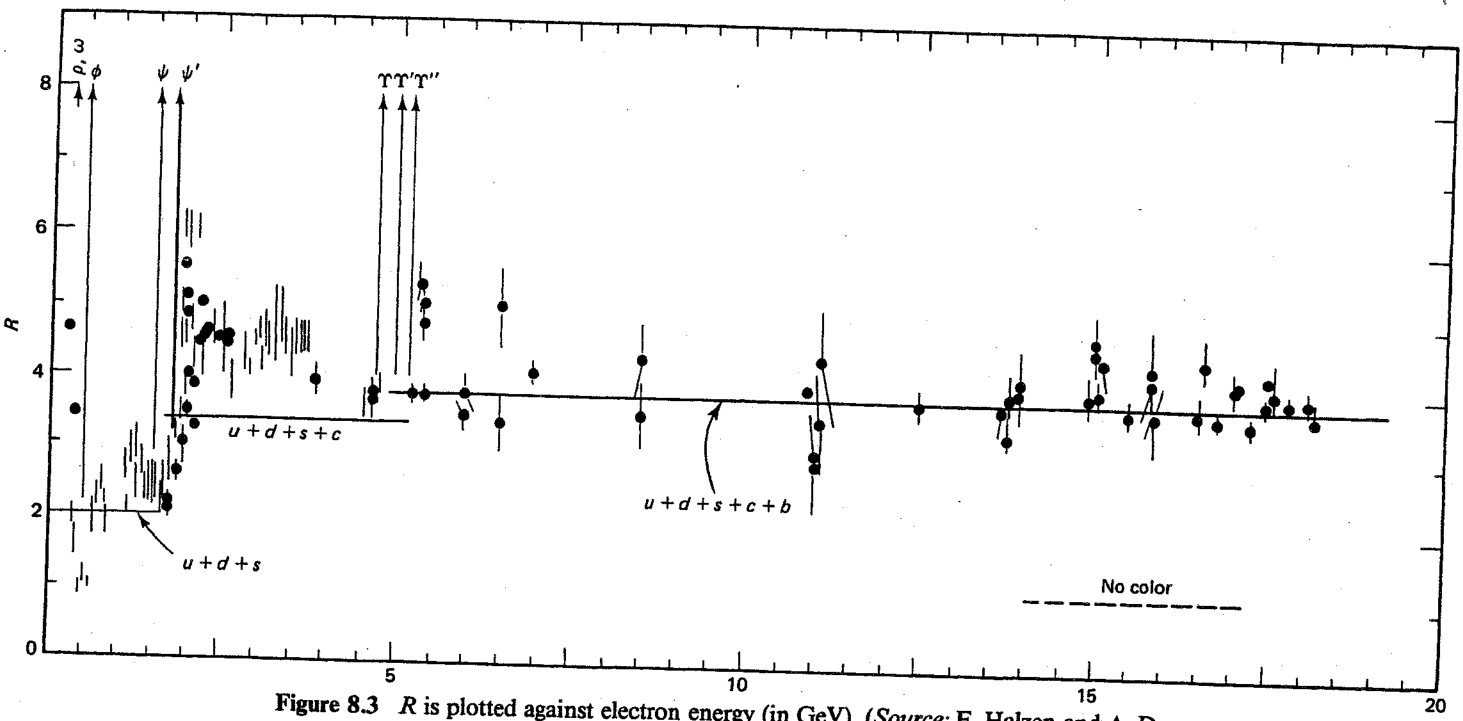



**Figure 8.3** $R$ is plotted against electron energy (in GeV). (*Source:* F. Halzen and A. D. Martin, *Quarks and Leptons* (New York: Wiley, copyright © 1984, p. 229. Reprinted by permission of John Wiley & Sons, Inc.)

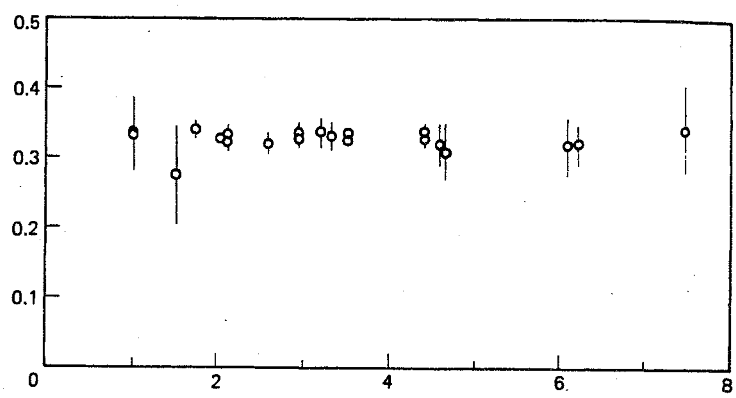
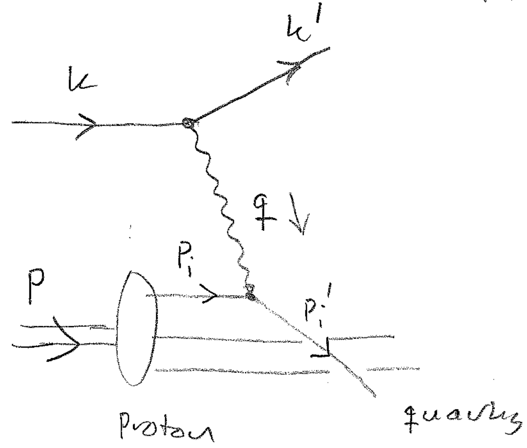


**Figure 8.6** Scaling behavior of the structure function $W_2$ in deep inelastic scattering. Here the quantity $-(q^2/2Mc^2x)W_2(q^2, x)$ is plotted against $-q^2$ (in GeV/$c$)$^2$, for $x = 0.25$. (*Source:* J. I. Friedman and H. W. Kendall; reproduced, with permission, from the Annual Review of Nuclear and Particle Science, Volume 22, © 1972 by Annual Reviews Inc.; page 227.)

Parton model assumptions

a) Assume that electron scatters elastically from a single quark in proton




b) Assume that quark (or parton) i carries a fraction $z_i$ of the proton momentum,

$$p_i = z_i P \qquad (\sum_i z_i = 1)$$

- Thus, $x_i = \dfrac{Q^2}{2q\cdot p_i} = \dfrac{x}{z_i}$

- With elastic scattering $x_i = 1 \Rightarrow z_i = x$

- Why $x_i = 1$ ?

$$q = (k-k') = (p_i' - p_i) \Rightarrow$$

elastic: $k\cdot p_i = k'\cdot p_i'$ ; $k'\cdot p_i = k\cdot p_i'$

$$x_i = \frac{-q^2}{2q\cdot p_i} = \frac{-(k-k')\cdot(p_i'-p_i)}{2(k-k')\cdot p_i} = \frac{k\cdot p_i + k'\cdot p_i' - k'\cdot p_i - k\cdot p_i'}{2(k-k')\cdot p_i}$$

$$= 1.$$

- Thus, smaller $x \rightarrow$ more inelastic!

- $x$ - momentum fraction of parton

- looks crazy when proton is at rest

  ($P = (M, \bar{0})$ ; $p_i = (xM, \bar{0}) \Rightarrow m_q = xM$ ?)

- Works fine when proton energy large, and using Lorentz-invariance can be also applied to rest frame

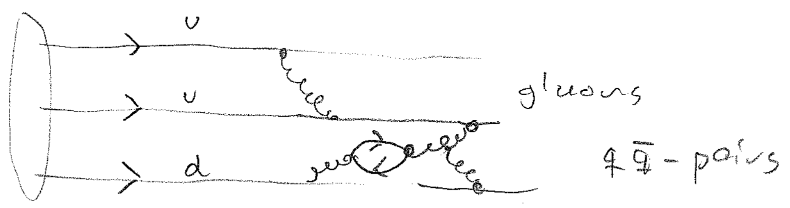
* It can be now shown, starting from $ep \to ep$ -scattering (e.g. page 149 (*)) and parton model assumptions, that Bjorken and Callan-Gross relations are valid

* Furthermore, we obtain

$$\overset{*}{\frac{x}{2}} F_1(x) = F_2(x) = x \sum_i Q_i^2 f_i(x) \qquad 0 \leq x \leq 1$$

where $f_i(x)$ is the probability distribution for parton type $i$ ($Q_i$ = charge)

* What the partons are



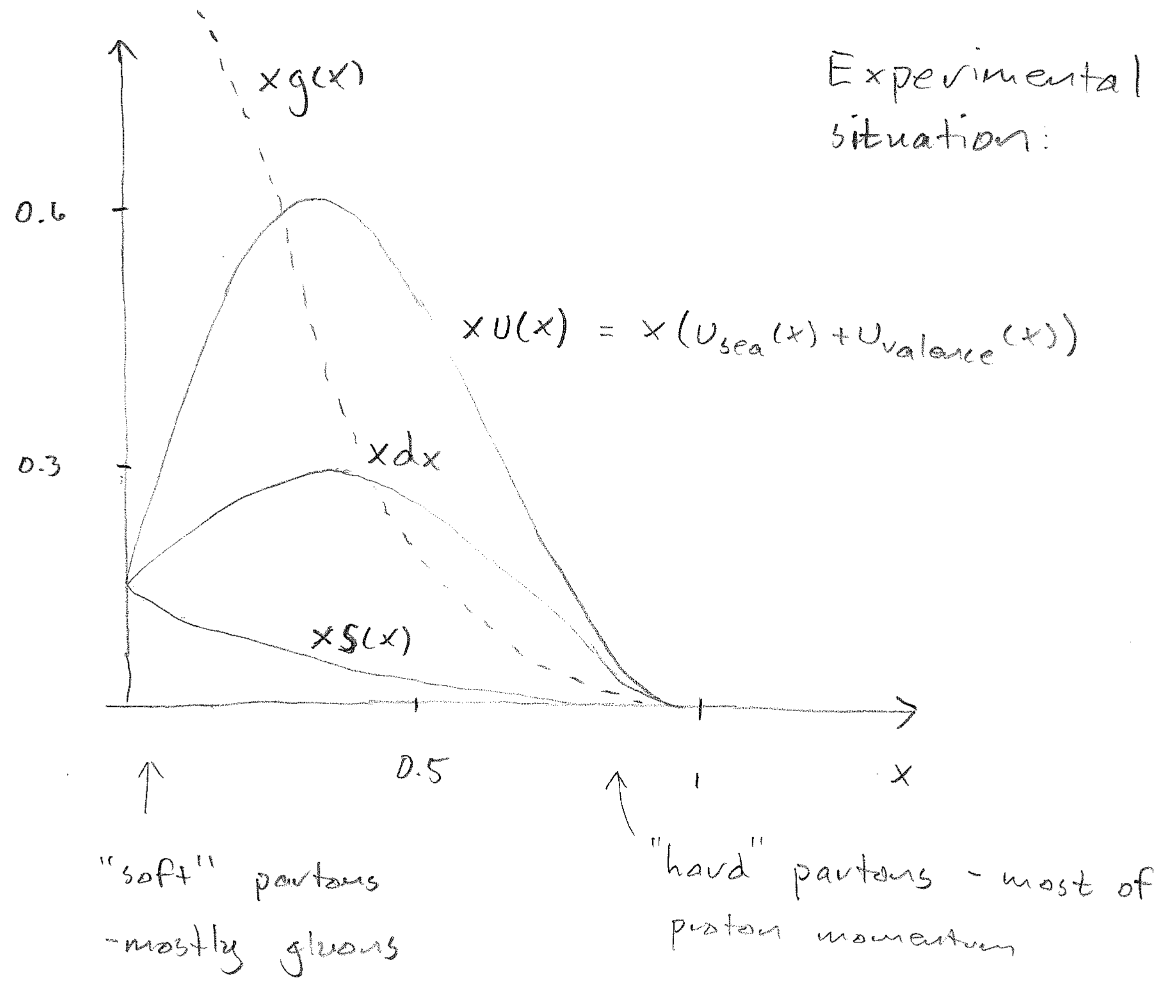

- "valence quarks": for proton, $uud$; the quark model content of proton

- "sea" quarks - virtual quark-antiquark pairs: $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ - depends on $x$

- gluons: $g(x)$

* Gluons have no charge; thus, they have no effect on $F_2(x)$. However, gluons carry a part of the total momentum

$$P = \int_0^1 dx \sum_i f_i(x) \cdot x P$$




* sea quark density ~same for all quarks,
$S(x) = \bar{S}(x) = U_{sea}(x) = \bar{U}_{sea}(x) = d_{sea}(x) = \bar{d}_{sea}(x)$

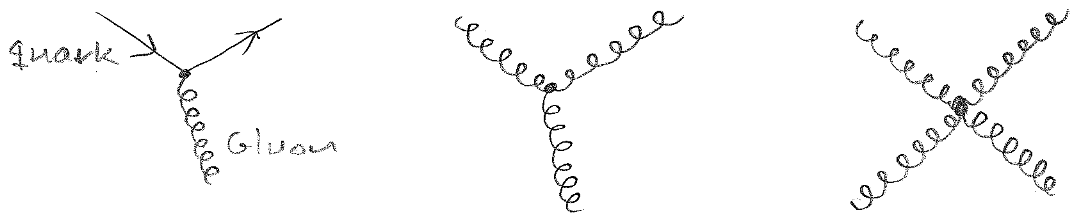
* For u, d - quarks, total distribution

$$u(x) = U_{sea}(x) + U_{valence}(x)$$

## 9.5 Quantum chromodynamics (QCD)

(Nambu, Gell-Mann, Fritzsch ; Gross, Wilczek, Politzer 1973)

- Proper theory of strong interactions (earlier versions were just incomplete models without proper dynamics)
- "Non-abelian gauge theory", Yang + Mills 1954
- Resembles quite a bit QED, but some complications
- Vertices:




3-gluon 4-gluon

* As opposed to QED, gluon interact with each other!

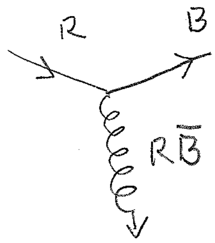
* 'Gauge invariance' demands that all vertices have the same coupling constant;


$g_s$ - strong (QCD) coupling constant

* Define $\alpha_s \equiv \frac{g_s^2}{4\pi}$. This is not small, typically $\alpha_s \gtrsim 1/10$.

- quarks carry colour $c = R, G, B$;
  gluons carry colour + anticolo

  Vertex:




- 3 colours → 3-component vector, $R = \begin{pmatrix}1\\0\\0\end{pmatrix}, G = \begin{pmatrix}0\\1\\0\end{pmatrix}, B = \begin{pmatrix}0\\0\\1\end{pmatrix}$

  ⇒ quarks transform as

  $$q^{c'} = U^{c'c} q^c \quad , \quad U \in SU(3) \qquad (U^\dagger U = 1, \ \det U = 1)$$

  ⇒ gluons belong to (see pages 53-58)

  $$3 \otimes \bar{3} = 1 \oplus 8$$

  gluons belong here
  → 8 gluon colours
  (8 Gell-Mann matrices, page 53)

  singlet
  $R\bar{R} + G\bar{G} + B\bar{B}$, does not exist.

  (substitute $u, d, s \to R, G, B$. From colour point of view, gluons are ~ "mesons" of the colour SU(3). However, do not push this too far! Gluons are massless elementary particles)

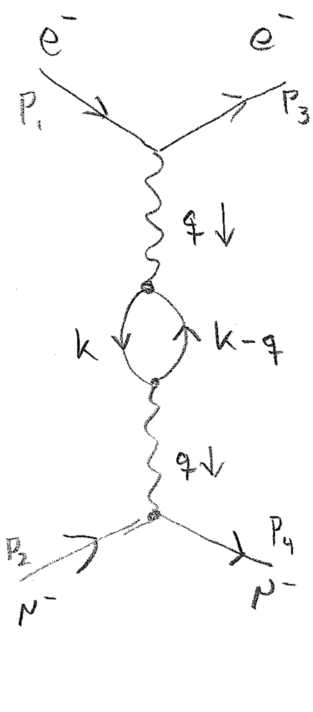
- Feynman rules complicated due to colour factors (extra combinatorics)
  (will not go in those; see Griffiths 280-284)

# Renormalisation and running coupling (QED & QCD)

- Already mentioned on page 9-11

* In QED and QCD the coupling constants ($e$ and $g_s$) are not really constant, but depend on energy

* In QED, consider $e^-\mu^- \to e^-\mu^-$ -scattering: at 4th order in $e$, there is "vacuum polarisation" diagram




- virtual photon splits into $e^+e^-$ -pair
- loop gives (neglecting $m_e$)

$$I_{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{Tr[\gamma_\mu \not{k} \gamma_\nu (\not{q}-\not{k})]}{k^2(q-k)^2}$$

- when $|k| \to \infty$, superficially this diverges as

$$\int_0^\infty dk\, k^3 \frac{k^2}{k^4} = \int_0^\infty dk \cdot k$$

* In reality, the contribution from the loop is log-divergent: (because of Tr and structure of the diagram)

$$I \simeq -e^2 \int^\infty dk \frac{1}{k} = -e^2 / \ln k$$

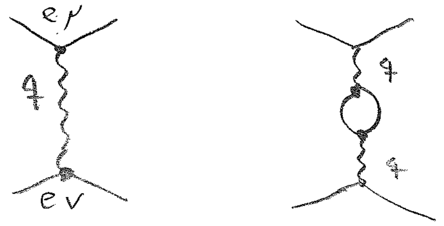
* let us regulate this by limiting $k^2 < \Lambda^2$: $\leftarrow \Lambda \to \infty$

(note: $-q^2 > 0$)

$$I_{\mu\nu} \to -e^2 \int^{k^2<\Lambda^2} \frac{d^4k}{(2\pi)^4} \cdots \simeq g_{\mu\nu} \frac{-e^2 q^2}{12\pi^2}\Big[\underbrace{\ln\frac{\Lambda^2}{\mu^2}}_{\text{divergent part}} - \underbrace{\ln\frac{|q^2|}{\mu^2}}_{\text{finite}}\Big]$$

* $\Lambda$ - cutoff energy scale; $\mu$ = renormalisation scale, a priori arbitrary

* Consider now the photon propagator + vertex (ignore $i$'s etc.)




$$e^2 \frac{g_{\mu\nu}}{q^2} \qquad\qquad e^2 \frac{g_{\mu\alpha}}{q^2}\frac{g_{\nu\beta}}{q^2} I^{\alpha\beta} = -e^2 \frac{g_{\mu\nu}}{q^2}\frac{e^2}{12\pi^2}\Big[\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{|q^2|}{\mu^2}\Big]$$

* Thus, total amplitude contains a part:

$$\mathcal{M} \propto \frac{e^2 g_{\mu\nu}}{q^2}\Big[1 - \frac{e^2}{12\pi^2}\Big(\ln\frac{\Lambda^2}{\mu^2} - \ln\frac{|q^2|}{\mu^2}\Big)\Big]$$

* Defining renormalised coupling

$$e_R = e\sqrt{1 - \frac{e^2}{12\pi^2}\ln\frac{\Lambda^2}{\mu^2}} \qquad \text{(depends on } \Lambda\text{, not } q^2\text{)}$$

we can write

subs $e \to e_R$ here, higher order

$$\mathcal{M} \propto \frac{e_R^2 g_{\mu\nu}}{q^2}\Big(1 + \frac{e_R^2}{12\pi^2}\ln\frac{|q^2|}{\mu^2}\Big) + \mathcal{O}(e^4)$$

* No sign of $\Lambda$ here! Divergence has been absorbed in $e_R$ → theory renormalisable

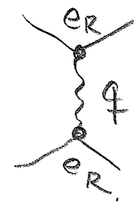
* $e_R$ is the physical coupling we observe: when we change $\Lambda$ we adjust the bare coupling $e$ so that $e_R$ constant

* $q^2$-dependence can be absorbed in $e_R$ too by defining

$$e_R(|q^2|) = e_R(\mu^2)\sqrt{1 + \frac{e_R^2(\mu^2)}{12\pi^2} \ln \frac{|q^2|}{\mu^2}}$$

$e_R(|q^2|)$ = running coupling

* Using $e_R(q^2)$, vacuum polarisation can be taken into account already at lowest order, just $e \to e_R(q^2)$:




* Effectively electron charge depends on energy ($q^2$)
In practise this is very small effect!

* $\mu^2$ chosen to be some relevant scale, e.g. $\mu^2 = m_e^2$ (in principle finite $m_e$ should be taken into account from the beginning)

* $e_R(|q^2|)$ <u>grows</u> when energy $(-q^2)$ increases

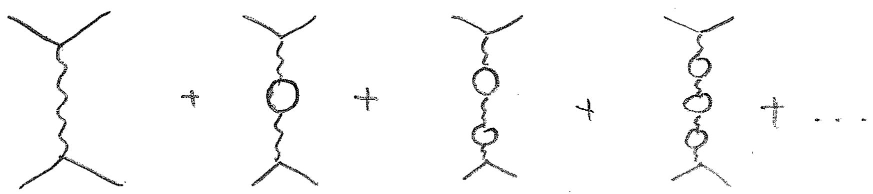
* Using $\alpha = \dfrac{e^2}{4\pi^2}$

$$\alpha_R(|q^2|) = \alpha\left(1 + \frac{\alpha}{3\pi}\ln\frac{|q^2|}{\mu^2}\right) \qquad (\mu^2 \to m_e^2)$$

where $\alpha = \alpha_R(\mu^2) = \frac{1}{137}$

This is valid for $-q^2 \gg m_e^2$

* More loops – these can be summed



$$\alpha \quad + \alpha A + \alpha A^2 + \alpha A^3 + \ldots \quad , \; A = \frac{\alpha}{3\pi}\ln\frac{|q^2|}{\mu^2}$$

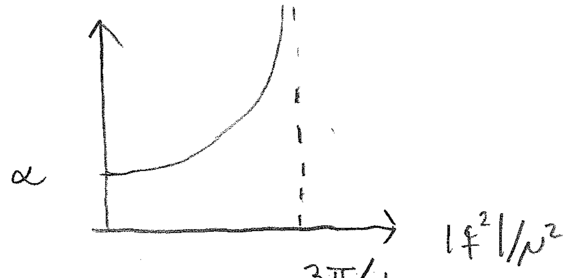
$$= \underline{\underline{\frac{\alpha}{1-A}}}$$

Thus, writing explicitely

$$\boxed{\alpha(|q^2|) = \frac{\alpha_0}{1 - \frac{\alpha_0}{3\pi}\ln\frac{|q^2|}{\mu^2}}}$$

$\alpha_0 = \alpha(\mu^2)$
$= \frac{1}{137}$, if $\mu^2$ small $\sim m_e^2$

* Conventional QED running coupling!




$e^{3\pi/\alpha} \simeq 10^{560}$ ; if $\mu^2 = m_e^2$, this is $|q| \sim 10^{270}$ MeV!

* Note - other charged fermions ($\mu, \tau$, quarks) contribute too $\Rightarrow$

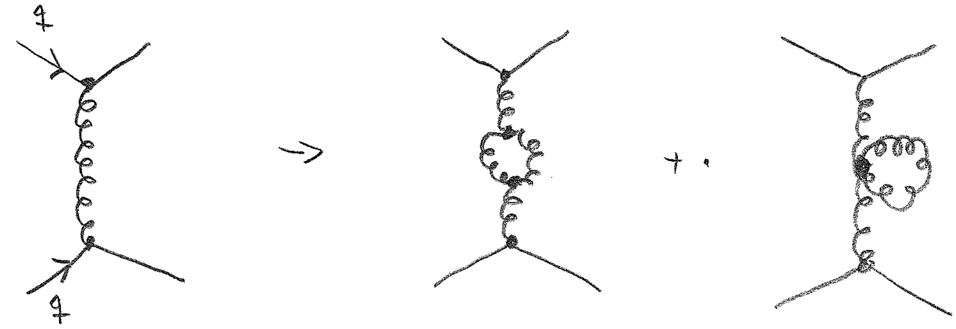
$$\alpha(|q^2|) = \frac{\alpha_0}{1 - C\frac{\alpha}{3\pi}\ln\frac{|q^2|}{\mu^2}}$$

$$C = \sum_{\substack{f \\ m_f^2 < |q^2|}}' Q_f^2 \quad ; \quad Q_f^2 = \underset{\text{leptons}}{1}, \underset{d,s,b}{\left(\tfrac{1}{3}\right)^2}, \underset{u,c,t}{\left(\tfrac{2}{3}\right)^2}$$

(only approximate, assumes $m_f \to 0$ for every fermion with $m_f^2 < |q^2|$)

* Scale $\mu$ is in principle arbitrary (just need to adjust $\alpha_0 = \alpha_0(\mu^2)$ to match physics)

* For <u>QCD</u>, we also have gluon loops



* however, boson loops → different sign w.r.t. fermion loops!

* Now

$$\alpha_s(|q^2|) = \frac{\alpha_s(\mu^2)}{1 + (11 \times N_c - 2N_f) \times \frac{\alpha_s(\mu^2)}{12\pi} \ln \frac{|q^2|}{\mu^2}}$$

$N_c = 3$ (3 colours)

$N_f$ = number of quarks with $m_q^2 < |q^2|$

* <u>Decreases</u> as $|q^2| \to \infty$ ⟶ <u>Asymptotic freedom</u>

* At high energies, QCD becomes weaker
  ⇒ quarks and gluons behave ~ free particles
  ⇒ Justification of parton model? "Free" partons within hadrons