3.3. Lie groups and generators

In physics the important groups ($SU(N)$) are Lie groups which can be parametrized with generators:

- Example: $SU(2)$

$$U = e^{i a_k \lambda_k} \quad , k = 1..3, \qquad \lambda_k = \frac{1}{2}\sigma_k \, , \, a_k \in \mathbb{R}$$

where Pauli σ-matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Because $\sigma_k^+ = \sigma_k$, clearly $U^+ = U^{-1}$ and $\det U = e^{\mathrm{Tr}(i a_k \lambda_k)} = e^0 = 1 \Rightarrow U \in SU(2)$

(in general $\det M = e^{\mathrm{Tr} \ln M}$)

Generators obey commutation relation:

$$[\lambda_i , \lambda_j] = i \, \epsilon_{ijk} \lambda_k$$

(More general for $SU(N)$: $[\lambda_i , \lambda_j] = i f_{ijk} \lambda_k$)

structure constant

- Generators of $SU(N)$ are hermitean ($\Leftarrow$ unitary) and traceless ($\det = 1$). $\Rightarrow$
- $SU(N)$ has $N^2 - 1$ generators (count real d.o.f.'s above!)

- Here Levi-Civita -symbols

$$\epsilon_{ijk} = \begin{cases} +1, & (ijk) = (123) \ \underline{\text{even}} \ \text{permutation} \\ -1, & (ijk) = (123) \ \text{odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

- Normalisation: $\underline{\mathrm{Tr}\, \lambda_i \lambda_j = \frac{1}{2} \delta_{ij}}$ ($SU(N)$)

- Infinitesimal transformations:

$$U \sim 1 + i a_k \lambda_k \quad , \quad |a_k| \ll 1$$

3.4 Rotation group

- Generated by angular momentum operator $\underline{\hat{J}_i = \epsilon_{ijk} \hat{r}_j \hat{p}_k}$, $i = x, y, z$

e.g.

$$J_z = (x p_y - y p_x)$$

- Commutator

$$\underline{[J_i, J_j] = i \epsilon_{ikl} J_l}$$

(Show component by component)

- Same as for $SU(2)$!

- Indeed, it can be shown that

$$\underline{\underline{\text{Rotation group} = SU(2)}}$$

(in 3d)

- Hermitean: $J_i^{+} = J_i$

- Casimir-operator is an operator formed from generators which commutes with all generators. It is easy to see that

$$\vec{J}^2 = \sum_{i=1}^{3} J_i^2$$

is Casimir; $[J^2, J_i] = 0$

- Thus, we can form simultaneous eigenstates of one J_i, say J_z, and J^2: (QM course)

$$\begin{cases} J_z |jm\rangle = m |jm\rangle \\ J^2 |jm\rangle = j(j+1)|jm\rangle \end{cases}$$

- Here $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$; $m = -j, \dots, j$: $2j+1$ different
- Vectors $|jm\rangle$ transform according to the irreducible $(2j+1)$-dim. representations of rotation group:

$$U(\vec{\theta})|jm\rangle = e^{-i\theta_k J_k}|jm\rangle$$

$$= \sum_{m'} d^j_{m'm}(\vec{\theta})\, |jm'\rangle$$

$(2j+1)^2$-dim. matrix, representation

- $d^j_{mm'}$ = $(2j+1)$-dim. representation of $SU(2)$

-

$j = 0$	$1/2$	1	j	= spin
dim 1	2	3	$2j+1$	

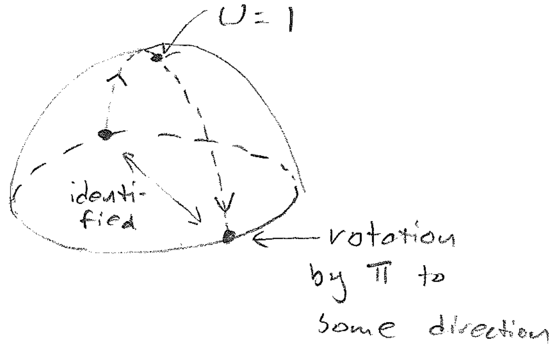
↑ trivial ↑ fundamental

Rotations of ordinary vectors, $SO(3)$

- Because of the connection $SU(2) \leftrightarrow$ rotations, representations of $SU(2)$ are commonly labeled by spin j, not by dimensionality as with $SU(N)$

Why $SU(2)$ = rotation group? Why spin $1/2$? (Extra material)

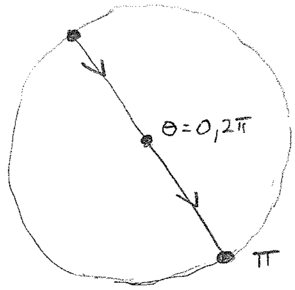
- Rotation group manifold is not simply connected: it is 3-dim surface of a sphere with "south" half cut off, and points at equator on opposite sides identified.



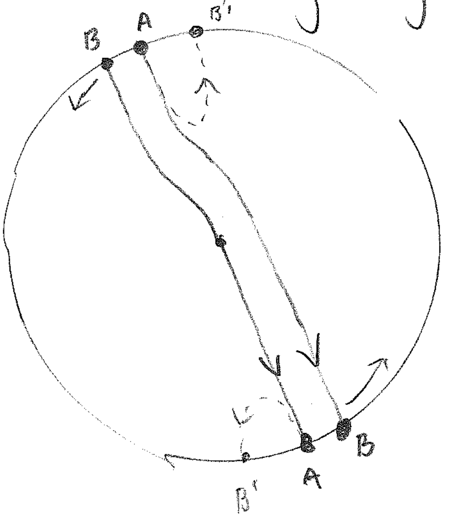
(2-dim "cartoon", reality 3-dim and impossible to draw)

- Full rotation (2π to some direction) cannot be shrunk to a point by continuous deformation

From above:



- However, going around twice can:



(Move point B close to A)

$\Rightarrow$ it is possible to have a continuous double-valued function which returns to original value only after rotation of 4π $\Rightarrow$ spin $\frac{1}{2}$ -rep.

For spin $\frac{1}{2}$, rotation by $2\pi \to$ mult. by -1

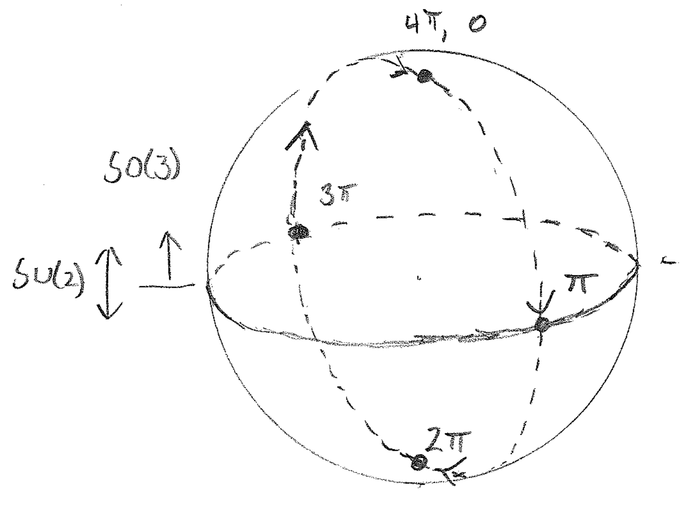
- What has this to do with $SU(2)$?

$SU(2)$ manifold is the surface of 4-sphere:

$$e^{-i\,\theta_k \tau_k} = \mathbb{1}\cos\theta - i\,\hat{\theta}_k \tau_k \sin\theta \qquad \theta = |\theta| \quad \hat{\theta} = \frac{\theta}{\theta}$$

$$= \mathbb{1}\,a_0 - i\,\tau_k\,a_k$$

with $\sum_{\mu=0}^{3} a_\mu^2 = 1.$



- $SU(2)$ is a simply connected extension ("covering group") of $SO(3)$
- All loops deformable to a point
- For integer spin representation (odd-dim. rep. of $SU(2)$) $U(2\pi) = U(0) = 1$
- For non-int. spin, (even-dim. representation of $SU(2)$) $U(2\pi) = -1$

End of extra $\uparrow$

- Thus, we can write

$$\langle jm' | U | jm \rangle = d^j_{m'm}$$

- and, for the generators

$$\langle jm' | \hat{J}_k | jm \rangle = (\tau^j_k)_{m'm}$$

$$\underline{d^j(\bar{\theta}) = e^{-i\theta_k \tau^j_k}}$$

- Simplest representation, $\underline{j=0}$: $d=1$ trivial rep.
  - Higgs boson, scalar mesons

- $\underline{j=1/2}$: 2-d fundamental rep. of $SU(2)$

$$\underline{|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \begin{pmatrix}1\\0\end{pmatrix}}; \qquad \underline{|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \begin{pmatrix}0\\1\end{pmatrix}}$$

$J_k = \tau_k = \frac{1}{2}\sigma_k$ Pauli σ-matrices

- $|\tfrac{1}{2}\,\tfrac{1}{2}\rangle$ and $|\tfrac{1}{2}\,-\tfrac{1}{2}\rangle$ are eigenstates of J_z. A general state is a linear combination of these. For example, eigenstates of

$$J_x = \frac{1}{2}\tau_x = \frac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad \text{(in this representation)}$$

are

$$\chi_\pm = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\\pm 1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(\begin{pmatrix}1\\0\end{pmatrix} \pm \begin{pmatrix}0\\1\end{pmatrix}\right)$$

with

$$J_x \chi_{\pm} = \pm\frac{1}{2}\chi_{\pm}$$

- Representations for generators at any j can be calculated using the operators

$$\hat{J}_{\pm} = \hat{J}_x \pm i\hat{J}_y \quad ; \quad J_x = \frac{1}{2}(J_+ + J_-) \, , \quad J_y = \frac{-i}{2}(J_+ - J_-)$$

which obey the relation

$$\hat{J}_{\pm}|jm\rangle = \sqrt{j(j+1) - m(m\pm 1)}\; |j, m\pm 1\rangle$$

- (Elementary) particles and spin:

	$s=0$	$s=1/2$	$s=1$	$s=3/2$	$s=2$	
Fundamental	Higgs boson	quarks leptons	$\gamma, g, W^{\pm}, Z$	–	(graviton)	
Composite	scalar mesons	baryon octet, $J=1/2$	vector mesons	baryon decouplet ($J=3/2$)	$J=2$ mesons	$\vec{J}=3,4$ mesons

(If SUSY is valid, then there should be a lot of extra particles, e.g.

SUSY	squarks sleptons	gauginos: $\tilde{\gamma}, \tilde{g}, \tilde{W}, \tilde{Z}$		gravitino		

)

3.5 Addition of angular momenta

- As in atomic physics, total spin of a hadron is a sum of its constituent spins + orbital angular momentum (with some caveats...)

- For example: spin of a meson

$$\bar{J} = \bar{J}_q + \bar{J}_{\bar{q}} + \bar{L}_{q\bar{q}} \quad ; \quad j_q = j_{\bar{q}} = \frac{1}{2}$$

- If $\bar{L} = 0$, quark-antiquark spins can be added so that these cancel ($j_{meson} = 0$) or add ($j_{meson} = 1$)

    - $j = 0$: pseudoscalar mesons
      $\pi, K, \eta, \eta' \ldots$

    - $j = 1$: vector mesons $\rho, K^*, \phi, \ldots$

- Thus, even though π^+ and ρ^+ contain $u\bar{d}$-quarks they are interpreted as different hadrons (as opposed to atomic physics)

- Different j-state $\Rightarrow$ different hadrons

- Adding 2 angular momenta:

$$|j_1\, m_1\rangle\,,\quad |j_2\, m_2\rangle$$

- z-component adds up directly: $\underline{m = m_1 + m_2}$

- Magnitude depends on relative orientation (which cannot be known in QM):

$$j = |j_1 - j_2|,\ |j_1 - j_2| + 1,\ \ldots,\ (j_1 + j_2)$$

- In general, with $m = m_1 + m_2$

$$|j_1\, m_1\rangle\, |j_2\, m_2\rangle = \sum_{j=|j_1-j_2|}^{j_1+j_2} C^{j,\, j_1,\, j_2}_{m_1,\, m_2}\, |j\, m\rangle$$

C: <u>Clebsch-Gordan</u> coefficients (see table), often denoted by

$$C^{j,\, j_1,\, j_2}_{m_1\, m_2} = \langle j,\, m_1 + m_2 \,|\, j_1\, j_2\, m_1\, m_2\rangle$$

- CG-coeff: probability amplitude of measuring $J^2 = j(j+1)$ in a system consisting of $|j_1 m_1\rangle$, $|j_2 m_2\rangle$

- Probability: $|\text{CG-coefficient}|^2$

20. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A $\sqrt{\ }$ is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

J M	J M	...
m_1 m_2	Coefficients	
m_1 m_2		
⋮		

1/2×1/2

m_1 m_2	J=1, M=+1
+1/2 +1/2	1

m_1 m_2	J=1, M=0	J=0, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=1, M=−1
−1/2 −1/2	1

1×1/2

m_1 m_2	J=3/2, M=+3/2
+1 +1/2	1

m_1 m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1 −1/2	1/3	2/3
0 +1/2	2/3	−1/3

m_1 m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0 −1/2	2/3	1/3
−1 +1/2	1/3	−2/3

m_1 m_2	J=3/2, M=−3/2
−1 −1/2	1

2×1

m_1 m_2	J=3, M=+3
+2 +1	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+2 0	1/3	2/3
+1 +1	2/3	−1/3

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/15	1/3	3/5
+1 0	8/15	1/6	−3/10
0 +1	6/15	−1/2	1/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1 −1	1/5	1/2	3/10
0 0	3/5	0	−2/5
−1 +1	1/5	−1/2	3/10

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0 −1	6/15	1/2	1/10
−1 0	8/15	−1/6	−3/10
−2 +1	1/15	−1/3	3/5

m_1 m_2	J=3, M=−2	J=2, M=−2
−1 −1	2/3	1/3
−2 0	1/3	−2/3

m_1 m_2	J=3, M=−3
−2 −1	1

1×1

m_1 m_2	J=2, M=+2
+1 +1	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+1 0	1/2	1/2
0 +1	1/2	−1/2

m_1 m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1 −1	1/6	1/2	1/3
0 0	2/3	0	−1/3
−1 +1	1/6	−1/2	1/3

m_1 m_2	J=2, M=−1	J=1, M=−1
0 −1	1/2	1/2
−1 0	1/2	−1/2

m_1 m_2	J=2, M=−2
−1 −1	1

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

2×1/2

m_1 m_2	J=5/2, M=+5/2
+2 1/2	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/5	4/5
+1 +1/2	4/5	−1/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1 −1/2	2/5	3/5
0 +1/2	3/5	−2/5

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0 −1/2	3/5	2/5
−1 +1/2	2/5	−3/5

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1 −1/2	4/5	1/5
−2 +1/2	1/5	−4/5

m_1 m_2	J=5/2, M=−5/2
−2 −1/2	1

3/2×1/2

m_1 m_2	J=2, M=+2
+3/2 +1/2	1

m_1 m_2	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/4	3/4
+1/2 +1/2	3/4	−1/4

m_1 m_2	J=2, M=0	J=1, M=0
+1/2 −1/2	1/2	1/2
−1/2 +1/2	1/2	−1/2

m_1 m_2	J=2, M=−1	J=1, M=−1
−1/2 −1/2	3/4	1/4
−3/2 +1/2	1/4	−3/4

m_1 m_2	J=2, M=−2
−3/2 −1/2	1

3/2×1

m_1 m_2	J=5/2, M=+5/2
+3/2 +1	1

m_1 m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2 0	2/5	3/5
+1/2 +1	3/5	−2/5

m_1 m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2 −1	1/10	2/5	1/2
+1/2 0	3/5	1/15	−1/3
−1/2 +1	3/10	−8/15	1/6

m_1 m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2 −1	3/10	8/15	1/6
−1/2 0	3/5	−1/15	−1/3
−3/2 +1	1/10	−2/5	1/2

m_1 m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2 −1	3/5	2/5
−3/2 0	2/5	−3/5

m_1 m_2	J=5/2, M=−5/2
−3/2 −1	1

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\, Y^m_\ell\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

2×3/2

m_1 m_2	J=7/2, M=+7/2
+2 +3/2	1

m_1 m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2 +1/2	3/7	4/7
+1 +3/2	4/7	−3/7

m_1 m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2 −1/2	1/7	16/35	2/5
+1 +1/2	4/7	1/35	−2/5
0 +3/2	2/7	−18/35	1/5

m_1 m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2 −3/2	1/35	6/35	2/5	2/5
+1 −1/2	12/35	5/14	0	−3/10
0 +1/2	18/35	−3/35	−1/5	1/5
−1 +3/2	4/35	−27/70	2/5	−1/10

m_1 m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1 −3/2	4/35	27/70	2/5	1/10
0 −1/2	18/35	3/35	−1/5	−1/5
−1 +1/2	12/35	−5/14	0	3/10
−2 +3/2	1/35	−6/35	2/5	−2/5

m_1 m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0 −3/2	2/7	18/35	1/5
−1 −1/2	4/7	−1/35	−2/5
−2 +1/2	1/7	−16/35	2/5

m_1 m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1 −3/2	4/7	3/7
−2 −1/2	3/7	−4/7

m_1 m_2	J=7/2, M=−7/2
−2 −3/2	1

3/2×3/2

m_1 m_2	J=3, M=+3
+3/2 +3/2	1

m_1 m_2	J=3, M=+2	J=2, M=+2
+3/2 +1/2	1/2	1/2
+1/2 +3/2	1/2	−1/2

m_1 m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2 −1/2	1/5	1/2	3/10
+1/2 +1/2	3/5	0	−2/5
−1/2 +3/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2 −3/2	1/20	1/4	9/20	1/4
+1/2 −1/2	9/20	1/4	−1/20	−1/4
−1/2 +1/2	9/20	−1/4	−1/20	1/4
−3/2 +3/2	1/20	−1/4	9/20	−1/4

m_1 m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2 −3/2	1/5	1/2	3/10
−1/2 −1/2	3/5	0	−2/5
−3/2 +1/2	1/5	−1/2	3/10

m_1 m_2	J=3, M=−2	J=2, M=−2
−1/2 −3/2	1/2	1/2
−3/2 −1/2	1/2	−1/2

m_1 m_2	J=3, M=−3
−3/2 −3/2	1

$$d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}}$$

$$d^1_{1,-1} = \frac{1-\cos\theta}{2} \qquad d^1_{0,0} = \cos\theta$$

2×2

m_1 m_2	J=4, M=+4
+2 +2	1

m_1 m_2	J=4, M=+3	J=3, M=+3
+2 +1	1/2	1/2
+1 +2	1/2	−1/2

m_1 m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2 0	3/14	1/2	2/7
+1 +1	4/7	0	−3/7
0 +2	3/14	−1/2	2/7

m_1 m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2 −1	1/14	3/10	3/7	1/5
+1 0	3/7	1/5	−1/14	−3/10
0 +1	3/7	−1/5	−1/14	3/10
−1 +2	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2 −2	1/70	1/10	2/7	2/5	1/5
+1 −1	8/35	2/5	1/14	−1/10	−1/5
0 0	18/35	0	−2/7	0	1/5
−1 +1	8/35	−2/5	1/14	1/10	−1/5
−2 +2	1/70	−1/10	2/7	−2/5	1/5

m_1 m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1 −2	1/14	3/10	3/7	1/5
0 −1	3/7	1/5	−1/14	−3/10
−1 0	3/7	−1/5	−1/14	3/10
−2 +1	1/14	−3/10	3/7	−1/5

m_1 m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0 −2	3/14	1/2	2/7
−1 −1	4/7	0	−3/7
−2 0	3/14	−1/2	2/7

m_1 m_2	J=4, M=−3	J=3, M=−3
−1 −2	1/2	1/2
−2 −1	1/2	−1/2

m_1 m_2	J=4, M=−4
−2 −2	1

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\,\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\,\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta+1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta - 1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta + 1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 20.1: Sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974). The signs and numbers in the current tables have been calculated by computer programs written independently by Cohen and at LBL.

Example. 1.

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\,|1\,-1\rangle = \sqrt{\tfrac{1}{3}}\,|\tfrac{3}{2}\,\tfrac{-1}{2}\rangle - \sqrt{\tfrac{2}{3}}\,|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle$$

A). Table $1 \times \frac{1}{2}$

B)

		3/2	1/2	J
m_1	m_2	-1/2	-1/2	M
0	-1/2	2/3	1/3	
-1	+1/2	1/3	-2/3	

$\longleftarrow$ find elements where $M = -\frac{1}{2}$

c) sqrt implied, e.g. $-\frac{2}{3} \rightarrow -\sqrt{\frac{2}{3}}$

Example 2:

Find the full decomposition of a state with 2 spin-$\frac{1}{2}$ particles:

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = |1\,1\rangle$$

$$|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\,|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\,0\rangle + \tfrac{1}{\sqrt{2}}|0\,0\rangle$$

$$|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle = \tfrac{1}{\sqrt{2}}|1\,0\rangle - \tfrac{1}{\sqrt{2}}|0\,0\rangle$$

$$|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle = |1\,-1\rangle$$

Thus, we have spin-1 (vector) state:

$$\begin{cases} |1\,1\rangle = |\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle \\ |1\,0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle + |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right) \\ |1\,-1\rangle = |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle \end{cases}$$

and a scalar spin-0 state:

$$|0\,0\rangle = \tfrac{1}{\sqrt{2}}\left(|\tfrac{1}{2}\,\tfrac{1}{2}\rangle|\tfrac{1}{2}\,\tfrac{-1}{2}\rangle - |\tfrac{1}{2}\,\tfrac{-1}{2}\rangle|\tfrac{1}{2}\,\tfrac{1}{2}\rangle\right)$$

- This encodes concretely the product of the representations,

$$2 \otimes 2 = 1 \oplus 3$$

↑ 2-dim spin-$\frac{1}{2}$-rep.

- Compact notation:

$$|j_1 m_1\rangle |j_2 m_2\rangle = |j_1 j_2 m_1 m_2\rangle$$

and

$$|j_1 j_2 m_1 m_2\rangle = \sum_{JM}{}' \underbrace{\langle JM | j_1 j_2 m_1 m_2\rangle}_{\text{Clebsch-Gordan coefficient}} |JM\rangle$$

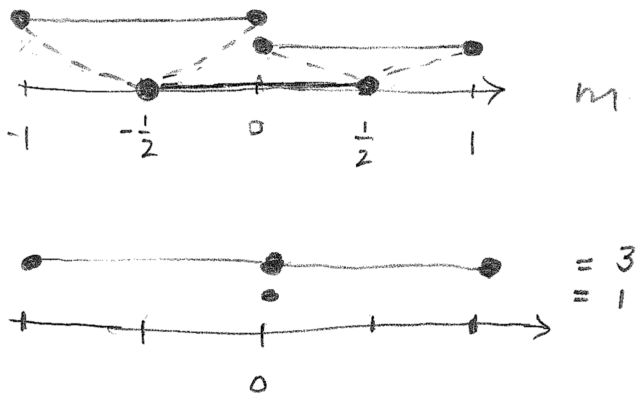
we also see that

$$|JM\rangle = \sum_{m_1 m_2}{}' \underbrace{\langle j_1 j_2 m_1 m_2 | JM\rangle}_{\text{same as } \langle JM | j_1 j_2 m_1 m_2\rangle,\ C\text{-}G \in \mathbb{R}.} |j_1 j_2 m_1 m_2\rangle$$

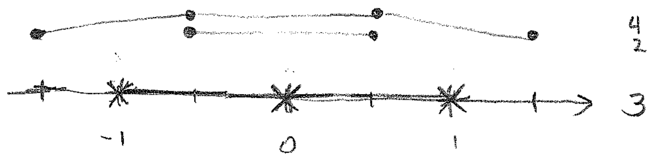
Thus, we can use C-G coefficients to both directions.

For SU(2), decomposition is easy to recognize graphically:

$2 \otimes 2 = 3 \oplus 1$



$2 \otimes 2 \otimes 2 = 2 \otimes (3 \oplus 1) = 2 \otimes 3 \oplus 2 = \underline{4 \oplus 2 \oplus 2}$



3 spin-$\frac{1}{2}$-quarks $\rightarrow$ spin-$\frac{1}{2}$, spin-$\frac{3}{2}$ hadrons

$\uparrow$

2 reps!

Thus, 1 =  $j=0$

2 =  $j=\frac{1}{2}$

3 =  $j=1$

etc.

This is rather obvious for SU(2), but will be more useful for SU(3)

3.6. Flavor symmetries

While spin is an exact quantum number due to rotational invariance, there are useful approximate symmetries.

One is $m_u \sim m_d \ll m_p, m_\pi$

Thus, switch $u \Leftrightarrow d$ $\Rightarrow$ mass of baryon does not change?

Indeed: $m_u \simeq 1$-5 MeV, $m_d \simeq 3$-9 MeV

$$\begin{cases} p: uud, & m_p = 938.28 \text{ MeV} \\ n: udd, & m_n = 939.57 \text{ MeV} \simeq m_p \end{cases}$$

Thus, if $m_u = m_d$ and we could neglect the EM interaction ($Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$), p and n would have exactly same mass

($m_p < m_n$ because $m_u < m_d$; el. charge acts to opposite direction)

If we denote $u = \begin{pmatrix}1\\0\end{pmatrix}$, $d = \begin{pmatrix}0\\1\end{pmatrix}$, we could take any combination $q = \alpha u + \beta d$; $|\alpha|^2 + |\beta|^2 = 1$, and still get same result

$\Rightarrow$ Isospin SU(2), basis $\begin{cases} u = |\frac{1}{2}\ \frac{1}{2}\rangle \\ d = |\frac{1}{2}\ {-\frac{1}{2}}\rangle \end{cases}$