3.7. Flavour SU(3)

- Exact if $m_u = m_d = m_s$ (not the case!)

$$\begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = U \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \qquad U \in SU(3)$$

- Generators of $SU(3)$: Gell-Mann matrices

$$\lambda_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \lambda_5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \lambda_6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda_7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad \lambda_8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

- $\begin{cases} \mathrm{Tr}\, \lambda_i \lambda_j = \frac{1}{2}\delta_{ij}, \quad \lambda_i^\dagger = \lambda_i \;\Rightarrow\; e^{-i\theta_a \lambda_a} \in SU(3) \\ \mathrm{Tr}\, \lambda_i = 0 \end{cases}$

- Note: now there are <u>2 commuting generators</u>, λ_3 and λ_8. Define (for flavour symmetry)

$\Rightarrow$ we can find eigenstates of <u>both</u> <u>simultaneously</u>.

- obviously
$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} = \overset{u}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}, \overset{d}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}, \overset{s}{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$
are eigenstates

$$\lambda_3 u = +\tfrac{1}{2} u; \qquad \lambda_3 d = -\tfrac{1}{2} d, \qquad \lambda_3 s = 0$$
$$\lambda_8 u = \tfrac{1}{2\sqrt{3}} u; \qquad \lambda_8 d = \tfrac{1}{2\sqrt{3}} d, \qquad \lambda_8 s = -\tfrac{2}{2\sqrt{3}} s$$

Thus, we can find:

λ_3: Isospin z-component I_3

λ_8: $\frac{\sqrt{3}}{2}(B+S) \equiv \frac{\sqrt{3}}{2} Y = \begin{cases} 1/2\sqrt{3} & u,d \\ -1/\sqrt{3} & s \end{cases}$

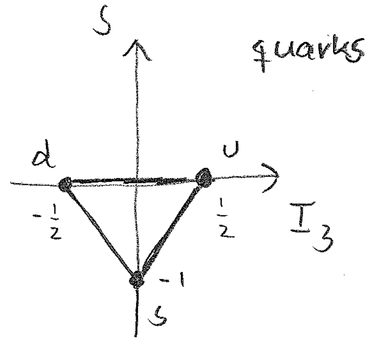
where B = baryon number ; S = strangeness

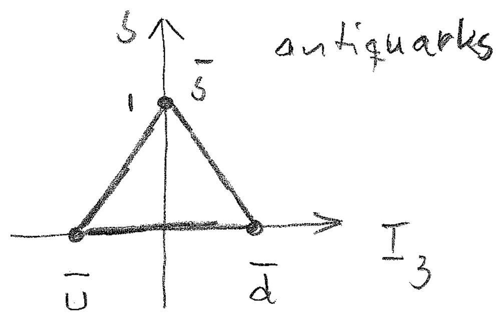
($S=0$ for u,d ; $S=-1$ for s)

- $Y = B+S$: <u>hypercharge</u>

- Thus, we can choose 2 quantum numbers, I_3, Y or I_3, S (matter of taste which base we take)

- 3 quarks form a triangle:





- Antiquarks have opposite I_3, S (or Y) quantum numbers $\Rightarrow$ inverted triangle

- Quarks transform according to fundamental rep. 3, antiquarks according to <u>conjugate</u> rep. $\bar{3}$!

- Note: I_3, S (and B and Y) <u>additive</u> quantum numbers

- Gell-Mann-Nishijima formula: electric charge

$$\underline{Q = I_3 + \tfrac{1}{2}(B+S) = I_3 + \tfrac{1}{2}Y}$$

(after 1974 c-quark was found and this should be modified accordingly... $I_3 + \frac{1}{2}(B+S+C)$)

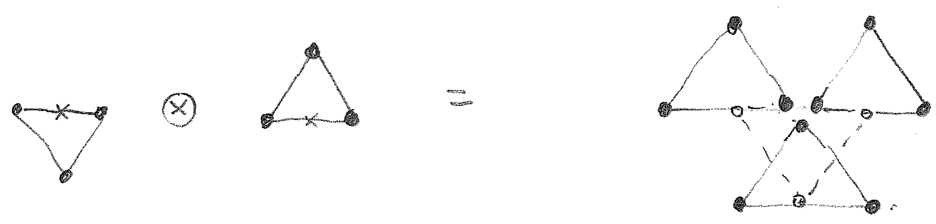
Why conjugate rep for $\bar{q}$?

- If $U \in SU(3)$, so does U^*
- $U \to U^* \Rightarrow \lambda_a \to -\lambda_a^*$
- Clearly, $-\lambda_8^* = -\lambda_8$ has <u>opposite</u> eigenvalues to λ_8 (likewise λ_3) $\Rightarrow$ opposite I_3, S.

  $\Rightarrow \begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$ transforms with U^*

- For $SU(2)$, conjugate rep $\hat{=}$ fundamental

- Additive $\to$ Reps can be combined <u>graphically</u>:
- E.g. $q\bar{q}$:



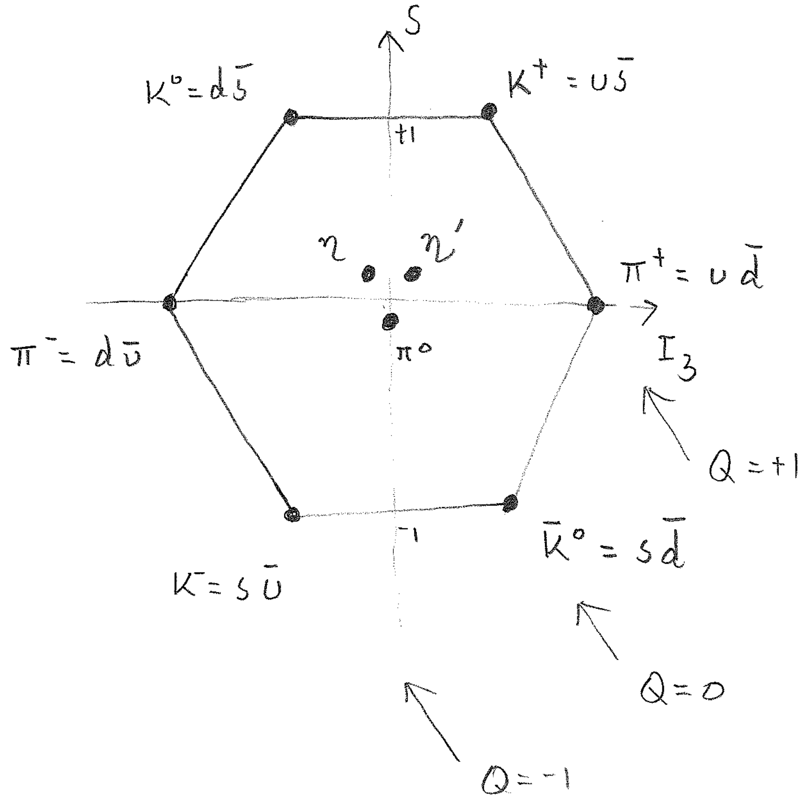
each corner $q\bar{q}$-state

MESONS:

- 9 states : $3 \otimes \bar{3} = 8 \oplus 1$

octet — singlet, SU(3)-symmetric

Thus, we obtain <u>pseudoscalar</u> <u>nonet</u> <u>$(s = \ell = 0)$</u>



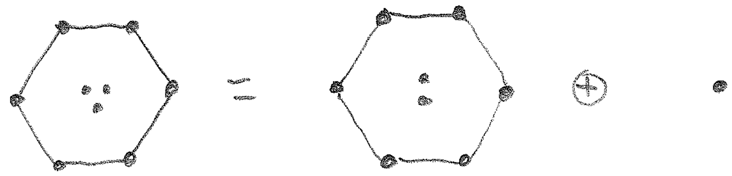
$$\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$$

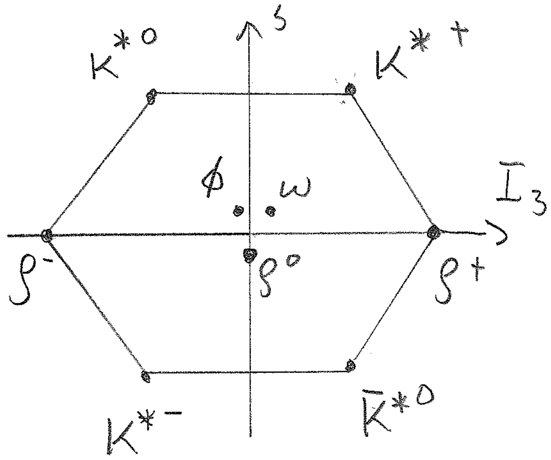
$$\eta' = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

- Compare to λ_a: $\begin{cases} K\text{'s}, \pi\text{'s}, \eta \Rightarrow \text{octet } 8 \text{ of } SU(3) \\ \eta' \Rightarrow \text{singlet } 1 \text{ of } SU(3) \end{cases}$

- If $m_u = m_d = m_s$, 8-mesons would have same mass $\neq \eta'$-mass.

Thus, really 

- If we combine $q\bar{q}$-pair in spin-1-state ($s=1, \ell=0$), we obtain <u>vector nonet</u>



$$\rho_0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$

$$\omega \cong \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d})$$

$$\phi \cong s\bar{s}$$

In this case ω, ϕ are not as should in $SU(3)$
– breaking of symmetry due to $m_s \gg m_u, m_d$

- <u>Masses</u>:

  - Masses equal within a given octet? <u>Broken</u> by <u>$m_u \neq m_s$</u>! We can derive <u>empirical mass formula</u> ($\ell=0$)

$$\boxed{M_{meson} = m_1 + m_2 + A\frac{\bar{S}_1 \cdot \bar{S}_2}{m_1 m_2}}$$

spin interaction, motivated by <u>hyperfine</u> interaction in atoms

- m_1, m_2: <u>constituent</u> quark masses (fit parameters)
- A: fit parameter
- $\bar{S}^2 = (\bar{S}_1 + \bar{S}_2)^2 = \bar{S}_1^2 + \bar{S}_2^2 + 2\bar{S}_1 \cdot \bar{S}_2 = 2 \cdot \frac{1}{2}(\frac{1}{2}+1) + 2\bar{S}_1 \cdot \bar{S}_2$

$$\Rightarrow \begin{cases} \text{pseudoscalar} & \bar{S}^2 = 0 \Rightarrow \bar{S}_1 \cdot \bar{S}_2 = -\frac{3}{4} \\ \text{vector} & \bar{S}^2 = 2 \Rightarrow \bar{S}_1 \cdot \bar{S}_2 = \frac{1}{4} \end{cases}$$

- Works surprisingly well:

		Meson	$m_{calculated}$	m_{true}	
$S=0$	octet	π	140	138	
		K	484	496	
		η	559	549	
	singlet	η'	349	958	$\leftarrow$!!
$S=1$		ρ	780	776	
		ω	780	783	
		K^*	896	892	No real octet/singlet at $S=1$
		ϕ	1032	1020	

where fit parameters

$$m_u = m_d = 310 \text{ MeV} \quad ; \quad m_s = 483 \text{ MeV}$$

$$A = (2m_u)^2 \cdot 160 \text{ MeV}$$

- These are <u>constituent</u> masses, not true quark masses = "effective" masses within hadron

- Spectacular failure at η'! Our naive treatment cannot explain this. More realistic picture is obtained using a new symmetry, <u>chiral symmetry</u>. It says: if

$$m_{u,d,s} \to 0 \,, \quad m_{\pi,K,\eta} \to 0 \quad \text{but } m_{\eta'} \sim \text{constant!}$$

BARYONS

- Similar SU(3) construction.

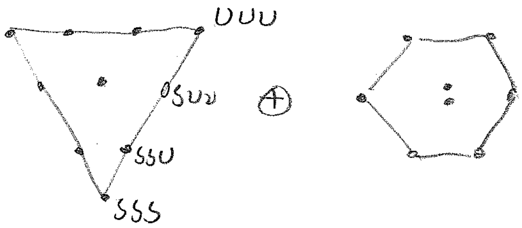
- $q \otimes q : \quad 3 \otimes 3 = 6 \oplus \bar{3}$

- $q \otimes q \otimes q = 3 \otimes 3 \otimes 3 = 6 \otimes 3 \oplus \bar{3} \otimes 3$

$= 10 \oplus 8 \oplus 8 \oplus 1$

6×3

$=$



$= 10 \otimes 8$

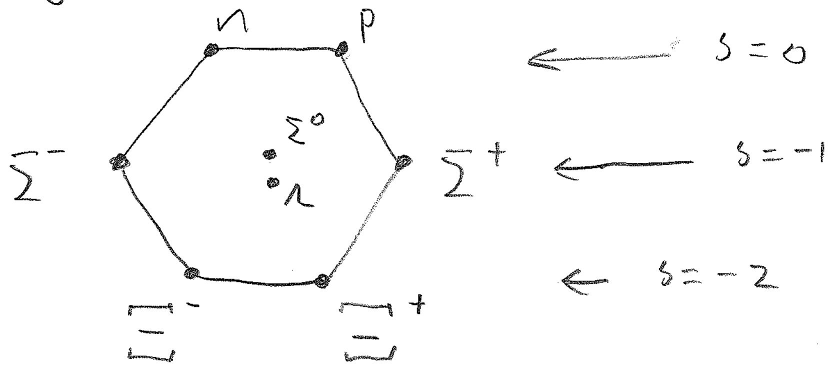
SU(3) has following reps:

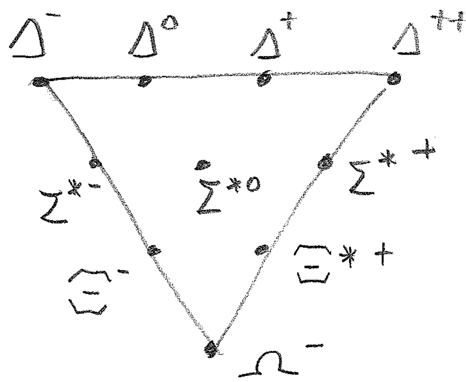
1, 3, 6, 8, 10, 15_a, 15_b, 24, 27, ...

$\bar{3}$, $\bar{6}$, $\overline{10}$, $\overline{15}_a$, $\overline{15}_b$, $\overline{24}$, ...

conjugates

Baryons:





(Gell-Mann predicted Ω^- in 1962, found 1964
$\rightarrow$ quark model "accepted")

- Where do these come from?
  - Consider vectors $A_a, B_b, C_c \in 3$ (i.e. 3-comp. vectors)

  - Now

$$A \otimes B = \overset{\text{9 comp.}}{A_a B_b} = \underbrace{\tfrac{1}{2}(A_a B_b + A_b B_a)}_{\substack{\text{symmetric} \\ 3 \cdot 4/2 = 6}} + \underbrace{\tfrac{1}{2}(A_a B_b - A_b B_a)}_{\substack{\text{antisymmetric} \\ 3 \cdot 2/2 = 3}}$$

i.e. $3 \otimes 3 = 6 \oplus \bar{3}$

(for $SU(N)$: $N \otimes N = \frac{N \times (N+1)}{2} \oplus \frac{N(N-1)}{2}$

$= 3 \oplus 1$, for $SU(2)$)

- symmetrize / antisymmetrize further:

$A \otimes B \otimes C = A_{\{a} B_b C_{c\}}$ fully symmetric $3 \cdot 4 \cdot 5 / 2 \cdot 3 = \underline{10}$

$+ A_{\{a} B_{[b\}} C_{c]}$ symmetric $A \leftrightarrow B$, antis. $B \leftrightarrow C$ (and $A \leftrightarrow C$) $\underline{8}$

$+ A_{[a} B_{\{b]} C_{c\}}$ symmetric $B \leftrightarrow C$, antis. $A \leftrightarrow B$ $\underline{8}$

$+ A_{[a} B_b C_{c]}$ fully antisymmetric $\underline{1}$

$= \epsilon_{abc} A_a B_b C_c$

• So, 10 : fully symmetric, 1 : antisymmetric
8 : mixed

EXTRA: SU(N) representations

This can be all codified in <u>Young tables</u> for SU(N):

$\square$ - fundamental, N

$\square\square$ - symmetrize, dim $N(N+1)/2$

$\begin{array}{c}\square\\\square\end{array}$ - antisymmetrize, $N(N-1)/2$

$N\left\{\begin{array}{c}\square\\\square\\\square\\\square\end{array}\right.$ - fully antisymmetric, N indices - <u>singlet 1</u>
($\epsilon_{abcd\ldots} A_a B_b \ldots$)

$(N-1)\left\{\begin{array}{c}\square\\\square\\\square\end{array}\right.$ - conjugate $\bar{N}$: $\underset{\substack{\uparrow\\\text{not summed}}}{\epsilon_{abc\ldots}} B_b C_c \ldots$ dim $= N$

General $\begin{array}{l}\square\square\square\square\\\square\square\square\\\square\end{array}$

• # rows $\leq N$
• # cols in row $n \geq$ cols in row $(n+1)$
• align to 1st column

- dimension (for SU(N))

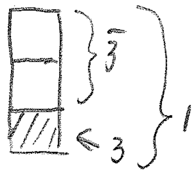
$$\frac{\begin{array}{|c|c|c|c|}\hline N & N+1 & N+2 & N+3\\\hline N-1 & N & N+1 & \\\hline N-2 & & & \\\hline\end{array}}{\begin{array}{|c|c|c|c|}\hline 7 & 4 & 3 & 1\\\hline 4 & 2 & 1 & \\\hline 1 & & & \\\hline\end{array}} = \frac{\Pi\,(\text{numbers})}{\Pi\,(\text{numbers})} = \text{dim}$$

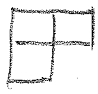
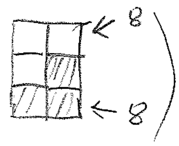
this line goes through $\overset{7}{4}$ squares

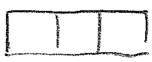
Thus, SU(3)

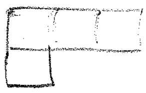
 : $\frac{\boxed{3}}{\boxed{1}} = 3$

 : $\frac{\boxed{3}\boxed{4}}{\boxed{2}\boxed{1}} = \frac{3\cdot 4}{2\cdot 1} = 6$

 : $\frac{\begin{array}{|c|}\hline 3\\ \hline 2\\ \hline\end{array}}{\begin{array}{|c|}\hline 2\\ \hline 1\\ \hline\end{array}} = 3$, rep $\bar{3}$ = 

 : $\frac{\begin{array}{cc}3 & 4\\ 2 & \end{array}}{\begin{array}{cc}3 & 1\\ 1 & \end{array}} = \frac{3\cdot 4\cdot 2}{3\cdot 1\cdot 1} = 8$ ($= \bar{8}$: )

 : $\frac{3\ 4\ 5}{3\ 2\ 1} = 10$

 : $\frac{3\ 4\ 5\ 2}{4\ 2\ 1\ 1} = 15$ etc

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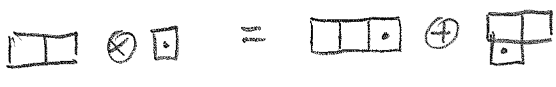
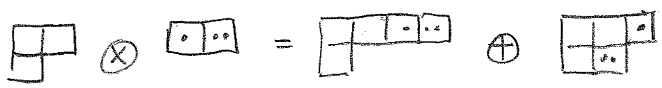
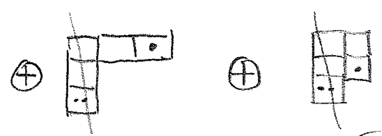
Product:

$3\otimes 3 =$  $= 6\oplus\bar{3}$

$6\otimes 3 =$  $= 10\oplus 8$

$8\otimes 6 =$ 



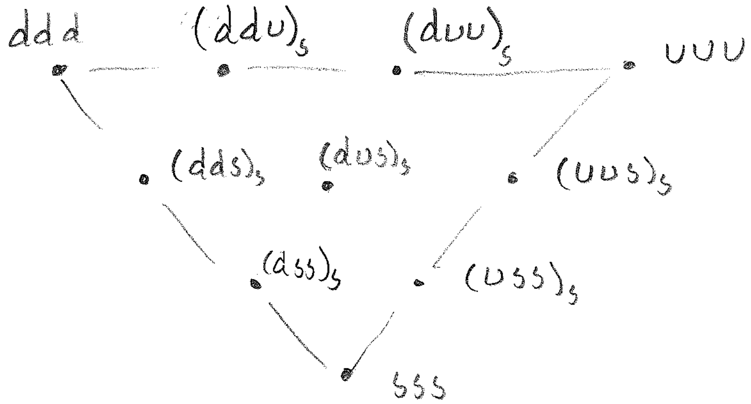
← full column = 1, i.e. $1\otimes\bar{3} = \bar{3}$

$= 24\oplus\overline{15}\oplus 6\oplus\bar{3}$

$$3 \times \bar{3} = \square \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix} = 8 \oplus 1$$

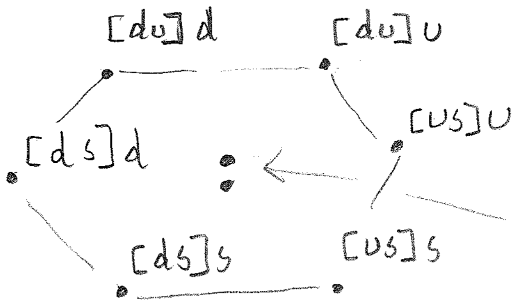
—— End of extra stuff ——

Thus, 10 is symmetric: $\left((abc)_s = abc + acb - bac - cba + bca + cab\right)$



1: $\bullet$ $(uds - usd - sdu - dus + sud + dsu) = [uds]$

8: antis. $1 \leftrightarrow 2$ $\qquad [ab] = ab - ba$



$[ds]u + [su]d - 2[ud]s$

$[ds]u - [su]d$

Another 8: antis. $2 \leftrightarrow 3$, for example

- Thus, baryons can belong to 3 reps: 10, 8 or 1 (antibaryons $\overline{10}$, 8, 1)

Symmetry of wave function

- quark quantum numbers
  - Color - 3 values, $SU(3)$ (3) $\begin{pmatrix} R \\ G \\ B \end{pmatrix}$
  - Flavor $SU(3)$ $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ (3)
  - Spin $SU(2)$ $s=1/2$ (2)

approximate! →

- Thus, when we combine quarks → hadrons, we have to take products of all reps:

$$\psi_{Hadron} = \psi_{space}\,\psi_{color}\,\psi_{Flavor}\,\psi_{spin}$$

↑ angular momenta

- Color: for physical states, has to be $SU(3)$ singlet!

Baryons: $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$;

Mesons: $3 \otimes \bar{3} = 8 \oplus 1$

- For baryons, singlet is fully antisymmetric

$$\psi_{color} = (RGB - RBG - GRB - BGR + GBR + GRB)$$
$$= \epsilon_{ABC}\, ABC$$

- Because quarks are fermions, total wave function <u>must be antisymmetric</u> with interchange of quarks

$\Rightarrow$ $\psi_{space}\,\psi_{flavor}\,\psi_{spin}$

must be symmetric!

- let us fix spatial ground state, $\ell = 0$ $(\ell' = 0)$
  $\uparrow$ light baryons

  $\Rightarrow$ $\underline{\psi_{space}}$ <u>symmetric</u>

$\Rightarrow$ $\psi_{Flav.}\,\psi_{spin}$ <u>symmetric</u>

Thus, we have:

- Flavor: $\begin{cases} 10 & \text{- symmetric} \\ 1 & \text{- antisymmetric} \\ 8 & \text{- mixed} \end{cases}$

- Spin: SU(2) $\underset{s=\frac{1}{2}}{2} \otimes \underset{\frac{1}{2}}{2} \otimes \underset{\frac{1}{2}}{2} = \underset{\frac{1}{2}}{2} \oplus \underset{\frac{1}{2}}{2} \oplus \underset{\frac{3}{2}}{4}$

$\left(\square \otimes \square \otimes \square = \square\square\square \oplus \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix} \oplus \begin{smallmatrix}\square & \square \\ \square & \end{smallmatrix} = 4 \oplus 2 \oplus 2 \right)$

- 4 fully symmetric, thus we have for decouplet

$$\psi_{10} = 10_{Flav} \otimes 4_{spin}$$

$\Rightarrow$ decouplet - baryons are spin $3/2$!