# 4. Relativistic quantum mechanics

(Griffiths 7.1 – 7.4)

## 4.1 Schrödinger eqn

- Non-relativistic:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \quad \text{Hamilton operator}$$

$$\boxed{i\partial_t |\psi\rangle = \hat{H} |\psi\rangle}$$

- $[\hat{x}_k, \hat{p}_l] = i\,\delta_{kl}$

- Eigenstates $\hat{H}|n\rangle = E_n|n\rangle$

$$\Rightarrow |n(t)\rangle = e^{-iE_n t}|n(0)\rangle$$

- Wave function

$$\psi(x,t) = \langle x|\psi(t)\rangle$$

$$\Rightarrow \left[-\frac{\nabla^2}{2m} + V(x)\right]\psi(x,t) = i\partial_t \psi(x,t)$$

- Harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$$

- $\hat{H}$ can be written in terms of creation- and annihilation operators
  - $\hat{a}$ annihilation
  - $\hat{a}^+$ creation

  $\hat{N} = \hat{a}^+\hat{a}$ number operator

  - $\hat{H} = \omega(\hat{N} + \frac{1}{2})$
  - $[\hat{a}, \hat{a}^+] = 1$ ; $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

    $\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$

## 4.2. Klein-Gordon eqn. (spin = 0)

- As we shall see, there is no "Schrödinger equation" for relativistic particles which would give us wave function + evolution in time. Need to use quantum field theory instead. However, for free particles we can write "classic" quantum mechanical wave equations.

- Remember: $p_i \to -i\partial_i \; ; \quad E \to i\partial_t$
  for Schrödinger wave equation, i.e.
  $$E = \frac{p^2}{2m} \; (+V) \quad \to \quad -\frac{\nabla^2}{2m}\psi = i\partial_t \psi$$

- Can this be done in relativistic case?

$$p_\mu p^\mu - m^2 = 0 \quad \Leftrightarrow \quad E^2 - \bar{p}^2 - m^2 = 0$$

$$\Rightarrow \boxed{\left[-\partial_t^2 + \vec{\nabla}^2 - m^2\right]\psi = 0} \qquad \text{Klein-Gordon eqn.}$$

- Describes free spin-0 particles (e.g. pions $\pi$)

Historically K-G eqn. was tought to be incorrect, because $\rho = |\psi|^2$ ("probability density") is not conserved ($\partial_t \rho \neq \nabla \cdot \vec{j}$), whereas
$$\rho' = i(\psi^* \partial_t \psi - \psi\, \partial_t \psi^*)$$
is conserved but not positive definite!
This is due to the 2nd time derivative.

This motivated Dirac to search for 1st order equation. (Actually, Schrödinger first considered K-G equation for QM before settling to the non-relativistic eqn.)

Presently these reservations are irrelevant: $\rho'$ is interpreted as charge density, or particles - antiparticles.

## 4.3. Dirac equation (spin-½)

- How to get 1st order in $\partial_t$? Factorize $p_\mu p^\mu - m^2$? If $\vec{p}=0$, this is easy:

$$(p^0)^2 - m^2 = (p^0 - m)(p^0 + m) = 0$$

$$\Rightarrow p^0 - m = 0 \quad \text{or} \quad p^0 + m = 0 \quad . \quad \text{ok.}$$

How about then

$$p_\mu p^\mu - m = (\beta^\mu p_\mu - m)(\gamma^\nu p_\nu + m) = 0$$

for some coefficients $\beta^\mu, \gamma^\nu$? Now

$$\begin{cases} \beta^\mu \gamma^\nu p_\mu p_\nu = g^{\rho\delta} p_\rho p_\delta & (*) \\ m\, p_\mu (\beta^\mu - \gamma^\mu) = 0 & \Rightarrow \beta^\mu = \gamma^\mu \end{cases}$$

Because (*) must be valid for all $p_\mu$, we must have

$$(\gamma^0)^2 = 1 \quad ; \quad (\gamma^i)^2 = -1 \qquad \begin{pmatrix} p = (p^0,0,0,0) \\ p = (0,p_1,0,0) \\ \cdots \end{pmatrix}$$

and

$$\sum_\mu{}' (\gamma^\mu)^2 p_\mu^2 + \sum_{\nu\neq\mu}{}' \gamma^\mu\gamma^\nu p_\mu p_\nu = \sum_\mu g^{\mu\mu} p_\mu^2$$

$$\Rightarrow \underline{(\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) = 0}, \quad \mu\neq\nu$$

$$\Rightarrow (\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu) \equiv \boxed{\{\gamma_\mu,\gamma_\nu\} = 2 g_{\mu\nu}}$$

- Dirac's brilliant idea (now practically obvious): use <u>matrices</u>
- Because $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$ $(\mu\neq\nu)$ $\Rightarrow$ $\mathrm{Det}\,\gamma_\mu\,\mathrm{Det}\,\gamma_\nu = (-1)^D\,\mathrm{Det}\,\gamma_\nu\,\mathrm{Det}\,\gamma_\mu$, thus, dimension $D$ of matrices must be even.
- At $D=2$ we don't find 4 matrices (recall $\{\sigma_i,\sigma_j\} = 2\delta_{ij}$ – 4th? )
- At $D=4$, we find

$$\gamma^0 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \quad ; \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

This choice is standard-representation. Here

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note: $\gamma_i = -\gamma^i$ ; $\gamma_0 = \gamma^0$

- Now we can take $\gamma^\mu p_\mu - m = 0$ ( $+m$ is equivalent!). Using $\underline{p_\mu \to i\partial_\mu}$ $[p^0 \to i\partial_0 ;\ p^i \to -i\partial_i]$

we obtain Dirac equation (1927)

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0}$$

$m \equiv m \cdot \mathbb{1}_{4\times 4}$

$\psi$ = Dirac spinor – 4-component vector

- One of the most important equs. in physics!
- Spin-$\frac{1}{2}$ particle has 2 degrees of freedom:

$\psi$ : 4 d.o.f.s : particles + antiparticles at the same time!

- Dirac first tought that anti-electron = proton. However, $m_{proton} \gg m_e$, whereas equ. indicates that $m_{anti} = m_{particle}$. Prediction for positron → found in 1932

(Dirac himself did not think his equ. was such of a big deal, because it was so easy to find...)

## 4.4. Spin-1 Maxwell equation

- Oldest relativistic field equation !
- 4-potential $A^\mu = (V, \vec{A})$

$$\Rightarrow \begin{cases} \vec{B} = \nabla \times \vec{A} \\ \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \end{cases}$$

- Define field strength tensor

$$F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & +B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- Maxwell equation (free, no charges)

$$\boxed{\partial_\mu F^{\mu\nu} = 0} \qquad \Big(\text{or } \partial_\mu F^{\mu\nu} = j^\nu = (\underbrace{\rho, \vec{j}}_{\text{4-current}})\Big)$$

$$\Rightarrow \underbrace{\partial_\mu \partial^\mu}_{} A^\nu - \partial_\mu \partial^\nu A^\mu = 0$$

$$\Box = \partial_0^2 - \nabla^2 \quad \text{d'Alembert operator}$$

- However, $A^\mu$ is not physical (observable), $\vec{E}$ and $\vec{B}$ (or equivalently $F^{\mu\nu}$) are.

Too many degrees of freedom

- Clearly, if $A^\mu$ is solution, so is
  $A^{\mu\prime} = A^\mu + \partial^\mu \phi$, because $F^{\mu\nu\prime} = \partial^\mu A^{\nu\prime} - \partial^\nu A^{\mu\prime} = F^{\mu\nu}$

  $\Rightarrow$ <u>Gauge transformation</u>, unphysical d.o.f.

- Additional condition, e.g. <u>Lorentz-gauge</u>:

$$\begin{cases} \partial_\mu A^\mu = 0 & \text{(Lorentz-gauge condition)} \\ \Box A^\mu = 0 & \text{(Maxwell equ.)} \end{cases}$$

  - Not complete, because there still is gauge freedom $A^{\mu\prime} = A^\mu + \partial^\mu \phi$, with $\Box \phi = 0$.

- <u>Coulomb gauge</u>:

$$\begin{cases} \Box A^\mu = 0 \\ A^0 = 0 \\ \nabla \cdot \vec{A} = 0 \end{cases}$$

  - Now remnant gauge freedom $\partial_0 \phi = \nabla^2 \phi = 0$ is static; i.e. does not change wave solutions.

## 4.5. Solution of Klein-Gordon eqn.

- In order to get to interactions, Feynman diagrams, etc. we need to discuss a bit field theory: Second Quantisation. We'll look at K-G:

$$(\partial_t^2 - \nabla^2 + m^2)\phi = (\partial_\mu \partial^\mu + m^2)\phi = 0$$

- Ansatz: $\phi = N e^{-ik_\mu x^\mu} = N e^{-ik^\mu x_\mu}$

$$\Rightarrow -k_\mu k^\mu + m^2 = 0 \Rightarrow \underline{k^0 = \pm\sqrt{\bar{k}^2 + m^2}}$$

- Thus, general solution $\phi(\bar{x}, t) = \phi(x)$ is a sum (use $E_{\bar{k}} = \sqrt{\bar{k}^2 + m^2}$)

$$\phi = \int d^3\bar{k} \left[ N_+(\bar{k}) e^{-iE_{\bar{k}}t + i\bar{k}\cdot\bar{x}} + N_-(\bar{k}) e^{+iE_{\bar{k}}t + i\bar{k}\cdot\bar{x}} \right]$$

switch $\bar{k} \to -\bar{k}$ here

$$= \int d^3\bar{k} \left[ N_+(\bar{k}) e^{-iE_{\bar{k}}t + i\bar{k}\cdot\bar{x}} + N_-(-\bar{k}) e^{+iE_{\bar{k}}t - i\bar{k}\cdot\bar{x}} \right]$$

Defining $\bar{k} \to \bar{p}$, $E_{\bar{k}} \to p^0 = E_{\bar{p}}$ ; $p\cdot p = m^2$

$$\Rightarrow E_k t - \bar{k}\cdot\bar{x} \to p\cdot x$$

And writing

$$\begin{cases} N_+(\bar{k}) \to \dfrac{1}{\sqrt{(2\pi)^3 2E_{\bar{p}}}}\, a_{\bar{p}} \\[2ex] N_-(-\bar{k}) \to \dfrac{1}{\sqrt{(2\pi)^3 2E_{\bar{p}}}}\, b^*_{\bar{p}} \end{cases}$$

we obtain

$$\phi(x) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[ a_{\bar{p}}\, e^{-ip\cdot x} + b^*_{\bar{p}}\, e^{ip\cdot x} \right] \qquad (*)$$

where $a_{\bar{p}}$, $b^*_{\bar{p}}$ are independent complex numbers if $\phi \in \mathbb{C}$; If $\phi \in \mathbb{R}$ $b_{\bar{p}} = a_{\bar{p}}$.

- Motivation for strange $\sqrt{\ }$ comes from <u>quantisation</u>
- Interpretation as particles ? (next)

$$e^{-ip\cdot x} = e^{-iE_{\bar{p}}t + i\bar{p}\cdot\bar{x}}$$

$$e^{i\bar{p}\cdot\bar{x}} = e^{iE_{\bar{p}}t + i\bar{p}\cdot\bar{x}}$$

$\leftarrow$ particle with negative energy $E = -E_{\bar{p}}$ ?

- Looks unphysical, $E < 0 = E_{vacuum}$ ?
- Another interpretation (Feynman, Stückelberg):

$$e^{+iE_{\bar{p}}t} = e^{-iE_{\bar{p}}(-t)}$$

= particle with $E > 0$, going backwards in time!

$\equiv$ <u>Antiparticle</u> going forward in $t$!

$\longrightarrow$ particle

$\longleftarrow$ antiparticle

Note: $(*)$ is just arbitrary rewriting of original solution, page 88. It becomes relevant when quantising.

## 4.6 Second quantisation

- Recall: classical mechanics $p = m\dot{x} = \dot{x} \quad (m=1)$
  $\longrightarrow$ quantum mechanics $[\hat{x}, \hat{p}] = i$

- Let us do the same here : $\begin{cases} x \to \phi(\bar{x},t) \\ p \to \partial_t \phi^*(\bar{x},t) \end{cases}$

i.e. different $(\hat{x}, \hat{p})$ pair at every spacetime location!

$\longrightarrow$ quantum field theory

$$[\hat{\phi}(\bar{x},t), \partial_0 \hat{\phi}^+(\bar{y},t)] = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

This leads to (compare harmonic oscillator)

$$\begin{cases} [\hat{a}_{\bar{p}}, \hat{a}^+_{\bar{p}'}] = \delta^{(3)}(\bar{p}-\bar{p}') \\ [\hat{b}_{\bar{p}}, \hat{b}^+_{\bar{p}'}] = \delta^{(3)}(\bar{p}-\bar{p}') \end{cases}$$

and $[\hat{a}_{\bar{p}}, \hat{a}_{\bar{p}'}] = [\hat{a}_{\bar{p}}, \hat{b}_{\bar{p}'}] = \ldots = 0$

where

$$\hat{\phi} \equiv \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[\hat{a}_{\bar{p}} e^{-ip\cdot x} + \hat{b}^+ e^{ip\cdot x}\right]$$

$$\hat{\phi}^+ = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \left[\hat{a}^+_{\bar{p}} e^{ip\cdot x} + \hat{b}\, e^{-ip\cdot x}\right]$$

(this explains the $\sqrt{\phantom{x}}$-normalisation factors)

Proof:

$$\partial_0 \hat{\phi}^\dagger(y) = \int \frac{d^3\bar{q}}{\sqrt{(2\pi)^3 2E_{\bar{q}}}} \cdot iE_q \left[\hat{a}^\dagger_{\bar{q}} e^{iq\cdot y} - \hat{b}_{\bar{q}} e^{-iq\cdot y}\right]$$

and

$$[\hat{\phi}(x), \partial_0 \hat{\phi}^\dagger(y)]_{y_0=x_0} = \int \frac{d^3\bar{p}\, d^3\bar{q}}{(2\pi)^3 2\sqrt{E_{\bar{p}}E_{\bar{q}}}} \cdot iE_q \Big[\underbrace{[\hat{a}_{\bar{p}}, \hat{a}^\dagger_{\bar{q}}]}_{\delta^{(3)}(\bar{p}-\bar{q})} e^{iq\cdot y - ip\cdot x} - \underbrace{[\hat{b}^\dagger_{\bar{p}}, \hat{b}_{\bar{q}}]}_{-\delta^{(3)}(\bar{p}-\bar{q})} e^{ip\cdot x - iq\cdot y}\Big]$$

$$= \int \frac{d^3\bar{p}}{(2\pi)^3} \frac{iE_{\bar{p}}}{2E_{\bar{p}}} \left\{ e^{-i\bar{p}\cdot(\bar{y}-\bar{x})} + e^{-i\bar{p}\cdot(\bar{x}-\bar{y})} \right\}$$

$$= i\int \frac{d^3\bar{p}}{(2\pi)^3} e^{i\bar{p}\cdot(\bar{x}-\bar{y})} = i\delta^{(3)}(\bar{x}-\bar{y}). \quad \square$$

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Thus, we go from classical field $\phi$ (with coefficients $a, b$) to field operator $\hat{\phi}$ by "upgrading" $a \to \hat{a}$, $b \to \hat{b}$, $a^* \to \hat{a}^\dagger$ etc.

Physical interpretation:

- $\hat{a}_{\bar{p}}$ annihilation operator, destroys particle with momentum $\bar{p}$. Thus, it must exist as an incoming particle
- $\hat{a}^\dagger_{\bar{p}}$ creation operator for particle, outgoing
- $\hat{b}_{\bar{p}}$ annihilation op. for antiparticle (incoming)
- $\hat{b}^\dagger_{\bar{p}}$ creation op. for antiparticle (outgoing)

- It can be shown that Hamilton-operator is

$$\hat{H} = \int d^3\bar{x} \left[ \partial_0 \hat{\phi}^+ \partial_0 \hat{\phi} + \nabla \hat{\phi}^+ \cdot \nabla \hat{\phi} + m \hat{\phi}^+ \hat{\phi} \right]$$

$$= \int d^3\bar{p}\, E_{\bar{p}} \Big[ \underbrace{\hat{a}^+_{\bar{p}} \hat{a}_{\bar{p}}}_{N^{\text{part.}}_{\bar{p}}} + \underbrace{\hat{b}^+_{\bar{p}} \hat{b}_{\bar{p}}}_{N^{\text{antipart.}}_{\bar{p}}} + \delta^{(3)}(0) \Big]$$

unphysical "vacuum energy"

- Or, conserved "charge"

$$\hat{Q} = \int d^3\bar{x} \left\{ \hat{\phi}^+ \partial_0 \hat{\phi} - \hat{\phi}\, \partial_0 \hat{\phi}^+ \right\} = \int d^3\bar{p} \left\{ \hat{a}^+_{\bar{p}} \hat{a}_{\bar{p}} - \hat{b}^+_{\bar{p}} \hat{b}_{\bar{p}} \right\}$$

$$= N^{\text{particle}} - N^{\text{antiparticle}} \sim \text{charge.}$$

- No negative energies here!

## 4.7. Solution of Maxwell equation

- Consider Coulomb gauge:

$$\begin{cases} \partial_\mu \partial^\mu A^\nu = 0 \\ A^0 = 0 \\ \partial_i A^i = 0 \end{cases} \qquad A^\mu \in \mathbb{R}$$

- Solution like K-G, except
  - $A \in \mathbb{R} \Rightarrow \hat{b}_{\bar{p}} = \hat{a}_{\bar{p}}$
  - 4 components $\mu = 0..3$, but not all physical. Introduce polarisation vector $\epsilon^\mu_{(\lambda)}$

$$\Rightarrow \hat{A}^\mu(\bar{x},t) = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \sum_\lambda{}' \epsilon^\mu_{(\lambda)}(\bar{p}) \left\{ \hat{a}^{(\lambda)}_{\bar{p}} e^{-ip\cdot x} + \hat{a}^{\dagger(\lambda)}_{\bar{p}} e^{ip\cdot x} \right\}$$

- Index $\lambda$ shows polarisation state. Gauge:

$$A^0 = 0 \Rightarrow \epsilon^0 = 0$$

$$\partial_i A^i = 0 \Rightarrow \epsilon^i \partial_i e^{-ip\cdot x} = \epsilon^i p_i e^{-ip\cdot x} = 0$$

$$\Rightarrow p_i \epsilon^i_{(\lambda)}(\bar{p}) = 0.$$

$\Rightarrow \bar{p}\cdot\bar{\epsilon} = 0$, i.e. polarisation perpendicular to $\bar{v}_{photon}$.

= free photon is "transverse"

$\Rightarrow \epsilon$ has 2 independent states, $\lambda = 1,2$.

$\Rightarrow$ 2 polarisation planes, or left- and right circular polarisation

<u>Completeness relation</u> (Coulomb gauge)

$$\sum_{\lambda=1,2} \epsilon^i_{(\lambda)}(\bar{p})\, \epsilon^j_{(\lambda)}(\bar{p}) = \delta^{ij} - \frac{p^i p^j}{\bar{p}^2}$$

(In different gauges looks different, but we shall use this)

- Example: if $\bar{p} = p\,\bar{e}_z$, we can choose
  $\epsilon_{(1)} = (0,1,0,0)$ ; $\epsilon_{(2)} = (0,0,1,0)$

  linear polarisation; or $\epsilon_{(\pm)} = \frac{1}{\sqrt{2}}(\epsilon_{(1)} \pm i\,\epsilon_{(2)})$

  circular polarisation

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- No antiphoton (no $\hat{b}$) because $A_\mu \in \mathbb{R}$ !
- $\hat{a}^{(\lambda)}_{\bar{p}}$ <u>annihilates</u> a photon of polarisation $\lambda$
  → must exist beforehand, incoming
- $\hat{a}^{(\lambda)+}_{\bar{p}}$ creates a photon → outgoing
- $\epsilon^\mu_{(\lambda)}(\bar{p})$, $\lambda = 1,2$ : polarisation

# 4.8 Solution of Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0 \quad ; \quad \psi \in \mathbb{C}^4$$

• Ansatz : $\psi = U(\bar{p})\, e^{-ip\cdot x}$ (plane wave)

$$\Rightarrow \boxed{(\gamma^\mu p_\mu - m)\, U(\bar{p}) = (\not{p} - m)\, U(\bar{p}) = 0}$$

where we use notation $\not{A} = \gamma^\mu A_\mu$

• Using $\gamma$-matrices from page 84, $\left[\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\right]$

$$\Rightarrow \begin{bmatrix} (p^0 - m)\mathbb{1} & -p^i\sigma^i \\ p^i\sigma^i & (-p^0 - m)\mathbb{1} \end{bmatrix} \begin{bmatrix} U_A \\ U_B \end{bmatrix} = 0 \qquad \left( p^i\sigma^i = \bar{p}\cdot\bar{\sigma} = \sum_{i=1}^{3} p^i\sigma^i \right)$$

$$\Rightarrow \begin{cases} U_A = \dfrac{1}{p^0 - m}\, \bar{p}\cdot\bar{\sigma}\, U_B \\ U_B = \dfrac{1}{p^0 + m}\, \bar{p}\cdot\bar{\sigma}\, U_A \end{cases} \Rightarrow U_A = \frac{1}{(p^0)^2 - m^2} (\bar{p}\cdot\bar{\sigma})(\bar{p}\cdot\bar{\sigma})\, U_A$$

Now $$p^i\sigma^i p^j\sigma^j = p^i p^j (\delta^{ij}\mathbb{1} + i\cancel{\epsilon^{ijk}}\sigma^k) = \bar{p}^2 \cdot \mathbb{1}$$

antisymm. $(i \leftrightarrow j)$

$$\Rightarrow \frac{\bar{p}^2}{(p^0)^2 - m^2} = 1 \Rightarrow (p^0)^2 = \bar{p}^2 + m^2 \Rightarrow p^0 = \pm E_{\bar{p}}$$

Solutions with positive and negative energy!

Again, these get particle - antiparticle (with $E>0$) interpretations

* Positive energy solution is written as
$\underset{\sim\sim\sim}{u(\bar{p})\, e^{-ip\cdot x}}$, and obeys $(\not{p}-m)u = 0$ (as above)

Because

$$(\not{p}-m)(\not{p}+m) = \gamma^\mu\gamma^\nu p_\mu p_\nu - m^2 = p^2 - m^2 = 0$$

we can imply that

$$u(\bar{p}) = (\not{p}+m)\cdot C \cdot u_0$$

spinor $u_0$ is independent of $\bar{p}$, and $C \in \mathbb{R}$.

- When $\bar{p} = 0$,

$$u(\bar{0}) = \begin{pmatrix} (p^0+m)\mathbb{I} & 0 \\ 0 & \underset{\overset{\shortparallel}{0}}{(-p_0+m)\mathbb{I}} \end{pmatrix} u_0 \times C = C\cdot 2m \begin{pmatrix} u_0^{(1)} \\ u_0^{(2)} \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ take $u_0$ to be $u_0 = \begin{pmatrix} \xi_s \\ 0 \end{pmatrix}$, where $s = \pm$ and

$$\xi_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \xi_- = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Thus

$$\underset{\sim\sim\sim}{u(\bar{p}, s) = C\cdot(\not{p}+m)\cdot\begin{pmatrix} \xi_s \\ 0 \end{pmatrix}}$$

* "Negative energy" solution $\underset{\sim\sim\sim}{v(\bar{p})\, e^{+ip\cdot x}}$, where now $p^0 = +E_{\bar{p}}$. This is solution of

$$(\not{p}+m)v(\bar{p}) = 0$$

and again we can write

$$v(\bar{p}) = C'(\not{p}-m)v_0, \quad C' \in \mathbb{R}$$

- $\bar{p} = 0 \Rightarrow v(\bar{0}) = -C'\, 2m \begin{pmatrix} 0 \\ 0 \\ v_0^{(3)} \\ v_0^{(4)} \end{pmatrix} \Rightarrow$ choose $v_0 = \begin{pmatrix} 0 \\ \xi_{-s} \end{pmatrix}$

$\Rightarrow$

$$\underset{\sim\sim\sim}{v(\bar{p}, s) = C'(\not{p}-m)\cdot\begin{pmatrix} 0 \\ \xi_{-s} \end{pmatrix}}$$