- u, v are <u>orthogonal</u>: $u^\dagger v = v^\dagger u = 0$.

- note: when $\vec{p} \neq 0$, all components non-zero!

<u>Normalisation:</u>

- Define <u>adjoint spinor</u> (Dirac-adjoint)

$$\bar{\psi} = \psi^\dagger \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \cdot \gamma^0$$

- It is useful to normalise u, v as follows:

$$\begin{cases} \bar{u}(\vec{p}, s)\, u(\vec{p}, s') = 2m\, \delta_{ss'} \\ \bar{v}(\vec{p}, s)\, v(\vec{p}, s') = -2m\, \delta_{ss'} \\ \bar{u} v = \bar{v} u = 0 \end{cases}$$

which determines the constants c, c':

$$c = -c' = \frac{1}{\sqrt{E_{\vec{p}} + m}}$$

Now also $u^\dagger u = v^\dagger v = 2 E_{\vec{p}} \cdot \delta_{ss'}$

Completeness relations

- In Feynman calculus we often need the following sums over spin states:

$$\begin{cases} \sum_s u_\alpha(\vec{p},s)\,\bar{u}_\beta(\vec{p},s) = (\not{p}+m)_{\alpha\beta} \\ \\ \sum_s v_\alpha(\vec{p},s)\,\bar{v}_\beta(\vec{p},s) = (\not{p}-m)_{\alpha\beta} \end{cases}$$

Proof: $u_\alpha \bar{u}_\beta = C^2\left[(\not{p}+m)\begin{pmatrix}\xi_s\\0\end{pmatrix}(\xi_s^T\ 0)(p_\mu\gamma^{\mu\dagger}+m)\gamma^0\right]_{\alpha\beta}$

Now $\sum_s \xi_s\xi_s^T = \begin{pmatrix}1\\0\end{pmatrix}(1\,0)+\begin{pmatrix}0\\1\end{pmatrix}(0\,1) = \mathbb{1}$

$$\Rightarrow \sum_s \begin{pmatrix}\xi_s\\0\end{pmatrix}(\xi_s^T\ 0) = \begin{pmatrix}\mathbb{1}&0\\0&0\end{pmatrix}$$

Also, it is easy to see that $\underline{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0}$,

thus,

$$\sum_s u\bar{u} = \frac{1}{p^0+m}(\not{p}+m)\begin{pmatrix}\mathbb{1}&0\\0&0\end{pmatrix}\gamma^0(\not{p}+m)$$

$$= \frac{1}{p^0+m}\begin{pmatrix}p^0+m & -\bar{p}\cdot\bar{\sigma}\\ \bar{p}\cdot\bar{\sigma} & -p^0+m\end{pmatrix}\begin{pmatrix}p^0+m & -\bar{p}\cdot\bar{\sigma}\\ 0 & 0\end{pmatrix}$$

$\sum_{ij} p_i\sigma_i p_j\sigma_j = \bar{p}^2\cdot\mathbb{1}$

$$= \frac{1}{p^0+m}\begin{pmatrix}(p_0+m)^2 & -(p^0+m)\,\bar{p}\cdot\bar{\sigma}\\ (p^0+m)\,\bar{p}\cdot\bar{\sigma} & -\bar{p}^2\end{pmatrix} \qquad -p_0^2+m^2 = (p_0+m)(-p_0+m)$$

$$= (\not{p}+m) \qquad \square$$

Physical interpretation

- $u(\vec{p},s)$ is the momentum space wave function of an particle
- $v(\vec{p},s)$ antiparticle

More precisely, with 2nd quantisation it can be shown

$u \leftrightarrow$ incoming particle
$v \leftrightarrow$ outgoing antiparticle
$\bar{u} \leftrightarrow$ outgoing particle
$\bar{v} \leftrightarrow$ incoming antiparticle

- Note that, for a "slow" particle u is dominated by upper 2 components:

$$u = \frac{1}{\sqrt{p^0+m}}(\not{p}+m)\begin{pmatrix}\xi\\0\end{pmatrix} = \frac{1}{\sqrt{p^0+m}}\begin{pmatrix}p^0+m & -\bar{p}\cdot\bar{\sigma}\\ \bar{p}\cdot\bar{\sigma} & (-p^0+m)\end{pmatrix}\begin{pmatrix}\xi\\0\end{pmatrix}$$

$$= \sqrt{p^0+m}\begin{pmatrix}\xi\\ \frac{\bar{p}}{p^0+m}\cdot\bar{\sigma}\,\xi\end{pmatrix} \leftarrow \text{small if } p^0 \gg |\bar{p}|$$

$\rightarrow$ non-relativistic limit with 2-comp. spinor = spin-1/2 Schrödinger eqn.

2nd quantisation

- In QFT it can be shown that

field operators are

$$\hat{\psi} = \int \frac{d^3\bar{p}}{\sqrt{(2\pi)^3 2E_{\bar{p}}}} \sum_{s=\pm1}^{1} \left[\hat{a}_{\bar{p}}^{(s)} u(\vec{p},s) e^{-ip\cdot x} + \hat{b}_{\bar{p}}^{+(s)} v(\vec{p},s) e^{+ip\cdot x} \right]$$

$$\hat{\psi}^{+} = \ldots\ldots \left[\hat{a}_{\bar{p}}^{+(s)} \bar{u}(\vec{p},s) e^{ip\cdot x} + \hat{b}_{\bar{p}}^{(s)} \bar{v}(\vec{p},s) e^{-ip\cdot x} \right]$$

Anticommutators

$$\{ \hat{a}_{\bar{p}}^{(s)}, \hat{a}_{\bar{p}'}^{+(s')} \} = \{ \hat{b}_{\bar{p}}^{(s)}, \hat{b}_{\bar{p}'}^{+(s')} \} = \delta^{(3)}(\bar{p}-p')\, \delta_{ss'}$$

$$\{ \hat{a}, \hat{a} \} = \{ \hat{a}, \hat{b} \} = \ldots = 0$$

Hamilton-op:

$$\hat{H} = \int d^3\bar{p}\, E_{\bar{p}} \sum_{s=\pm1}^{1} \left[\hat{a}_{\bar{p}}^{+(s)} \hat{a}_{\bar{p}}^{(s)} + \hat{b}_{\bar{p}}^{+(s)} \hat{b}_{\bar{p}}^{(s)} - \delta^{(3)}(0) \right]$$

0-point energy

Charge

$$\hat{Q} = \int d^3\bar{x}\, \hat{\bar{\psi}} \gamma_0 \hat{\psi}$$

$$= \int d^3\bar{p} \sum_{s} \left[\hat{a}_{\bar{p}}^{+(s)} \hat{a}_{\bar{p}}^{(s)} - \hat{b}_{\bar{p}}^{+(s)} \hat{b}_{\bar{p}}^{(s)} \right] = N_{part.} - N_{antip.}$$

Physically:

$\hat{a} \leftrightarrow u \rightarrow$ incoming particle

$\hat{a}^{+} \leftrightarrow \bar{u} \rightarrow$ outgoing particle

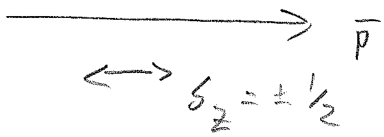
$\hat{b} \leftrightarrow \bar{v} \rightarrow$ incoming antiparticle

$\hat{b}^{+} \leftrightarrow v \rightarrow$ outgoing antiparticle

- Note: $\psi^\dagger$ is not used much, because $\psi^\dagger \psi$ is not Lorentz invariant $(u^\dagger u = 2E)$
- On the other hand, $\bar{\psi}\psi$ is $(\bar{u}u = 2m)$

Helicity

- "spin along the direction of motion"



- direction of motion $\bar{e}_{\bar{p}} = \bar{p}/|\bar{p}|$
- Helicity: $h_{\bar{p}} = \vec{e}_{\bar{p}} \cdot \vec{\Sigma}$, $\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$
- Spin along $\bar{e}_{\bar{p}}$: $s_{\bar{p}} = \frac{1}{2} h_{\bar{p}}$
- Now $h^2 = \mathbb{I} \Rightarrow$ eigenvalues ± 1
- For $\bar{p} = (0,0,p)$, from p. 99 we see that $h_{\bar{p}}\, u(\bar{p}, s) = s\, u(\bar{p}, s)$, or helicity is just s !

Chivality

- Chivality operator $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$
- $\gamma_5^2 = \mathbb{1} \Rightarrow$ eigenvalues ± 1
- $\{\gamma_5, \gamma_\mu\} = 0$
- "Left" and "right" projectors

$$\underline{P_L = \frac{1-\gamma_5}{2}} \quad ; \quad \underline{P_R = \frac{1+\gamma_5}{2}}$$

$* \; P_L^2 = P_L \; ; \quad P_R^2 = P_R \; ; \quad P_L P_R = P_R P_L = 0$

$P_L + P_R = 1$

$\Rightarrow$ every state can be decomposed as

$$u = (P_L + P_R)\, u = u_L + u_R$$

- <u>Weak interactions</u> only with P_L-component, i.e. left-handed spinors $P_L u = u_L$!
(to be discussed later)

- If $m=0$, chivality $\Leftrightarrow$ helicity:

$\bar{p} = (0, 0, |\bar{p}|)$, $m=0$

$$\Rightarrow u(\bar{p}, s) = \sqrt{|\bar{p}|} \begin{pmatrix} \xi \\ \sigma_z \xi \end{pmatrix}$$

$$\Rightarrow \underline{\gamma_5 u} = \sqrt{|\bar{p}|} \cdot \begin{pmatrix} \sigma_z \xi \\ \xi \end{pmatrix} = \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} u = \underline{h_{\bar{p}}\, u}$$

Neutrinos are only left-handed:

$P_R u_\nu = 0 \quad \Rightarrow \quad h = -1$

4.9. Creation and annihilation ops and eigenstates

- Vacuum $|0\rangle$; $\langle 0|0\rangle = 1$
- $\hat{a}\,|0\rangle = 0$; $\langle 0|\,\hat{a}^\dagger = 0$

  (same with $\hat{b}$)

- Momentum eigenstate $|\phi_{\bar{p}}\rangle = \hat{a}^\dagger_{\bar{p}}|0\rangle$
- Normalisation: $\langle \phi_{\bar{p}}| = \langle 0|\,\hat{a}_{\bar{p}}$

$$\langle \phi_{\bar{p}'}|\phi_{\bar{p}}\rangle = \langle 0|\,\hat{a}_{\bar{p}'}\,\hat{a}^\dagger_{\bar{p}}|0\rangle$$

$$= \langle 0|[\hat{a}_{\bar{p}'},\hat{a}^\dagger_{\bar{p}}]|0\rangle = \delta^{(3)}(\bar{p}-\bar{p}')$$

- This implies $\langle \phi_{\bar{p}}|\phi_{\bar{p}}\rangle = \delta^{(3)}(\bar{0})$?

How to understand this? Recall

$$\delta(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, e^{ipx}$$

restrict x to $[-L/2, L/2]$

$$\simeq \frac{1}{2\pi}\int_{-L/2}^{L/2} dx\, e^{ipx} \quad \Rightarrow \quad \delta(0) \mathrel{\hat{=}} \frac{L}{2\pi}$$

Thus, at finite V, $\delta^{(3)}(0) \to \dfrac{V}{(2\pi)^3}$