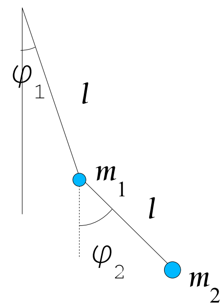
1. Consider a coplanar double pendulum shown in figure.



The Lagrangian of this pendulum is

$$
\begin{aligned}
L &= \frac{1}{2}(m_1+m_2)l^2\dot{\varphi}_1{}^2 + \frac{1}{2}m_2 l^2\dot{\varphi}_2{}^2 + m_2 l^2\dot{\varphi}_1\dot{\varphi}_2\cos(\varphi_1-\varphi_2) \\
&\quad +(m_1+m_2)gl\cos\varphi_1 + m_2 gl\cos\varphi_2.
\end{aligned}
$$

Show that the frequencies of small oscillations are

$$\omega^2 = \frac{g}{m_1 l}\left(m_1+m_2 \pm \sqrt{m_2(m_1+m_2)}\right).$$

2. Consider a system of masses in a plane concatenated to each other with springs. Assume that these masses can vibrate both in the longitudinal and in the transversal directions at the plane.
Show that the Lagrangian for small oscillations is

$$L = \sum_{i=1}^{n}\frac{1}{2}m(\dot{\eta}_i{}^2+\dot{\mu}_i{}^2) - \sum_{i=0}^{n}\left[\frac{k}{2}(\eta_{i+1}-\eta_i)^2 + \frac{\tau}{2a}(\mu_{i+1}-\mu_i)^2\right] + C,$$

where C is constant, η_i and μ_i are the displacements of masses from their equilibrium positions in longitudinal and transversal directions, k is the spring constant and $\tau = k(a-a_0)$ is the tension force at equilibrium (We assume that the springs between masses are stretched at equilibrium so that their lengths are a. Parameter a_0 is the length of an unstretched spring.).
Show that the longitudinal and transversal oscillations happen independently.
Write down the dispersion relation for both independent oscillations and use the dispersion relation to obtain the propagation velocities $c_\parallel$ and $c_\perp$ for long wavelenghts.

(Hint: As an intermediate step use

$$V = \frac{k}{2}\sum_{i=0}^{n}[\sqrt{(\eta_{i+1}-\eta_i+a)^2+(\mu_{i+1}-\mu_i)^2}-a_0]^2.$$

Be careful to keep all second order terms in η.)

3. Consider a two-dimensional square array of $2N$ massless strings of unperturbed length $(N+1)a$ with fixed endpoints. The strings are stretched to tension τ and have point masses m located at each intersection (ia, ja), where $1 \leq i, j \leq N$. Assume that masses can vibrate only in the transversal direction (direction perpendicular to the plane of the array).

a) Argue that the Lagrangian for small transverse displacements η_{ij} of the masses is

$$L = \frac{m}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\dot{\eta_{ij}}^2 - \frac{\tau}{2a}\left[\sum_{i=0}^{N}\sum_{j=1}^{N}(\eta_{i+1j}-\eta_{ij})^2+\sum_{i=1}^{N}\sum_{j=0}^{N}(\eta_{ij+1}-\eta_{ij})^2\right].$$

b) Construct the equations of motion for travelling waves with the wave vector $\vec{k} = k_x\hat{x} + k_y\hat{y}$, and show that the dispersion relation is

$$\omega = \sqrt{\frac{4\tau}{am}(\sin^2\frac{k_x a}{2}+\sin^2\frac{k_y a}{2})}.$$

c) Show that the dispersion relation for long wave lengths is isotropic (same in all directions).