

1. Consider a Lagrangian density, which is expressed in the form

$$\mathcal{L} = \mathcal{L}\left(\psi, \nabla\psi, \frac{\partial\psi}{\partial t}, \psi^*, \nabla\psi^*, \frac{\partial\psi^*}{\partial t}; x, t\right),$$

where $\psi = a + ib$ is a complex function, whose real and imaginary parts are independent field variables.

Show that the equations of motion for a and b are equivalent with the equations of motion, which one obtains by simply taking ψ and ψ^* as independent field variables.

Let's now consider the Lagrangian density

$$\mathcal{L} = -\frac{\hbar^2}{2m} \nabla\psi \cdot \nabla\psi^* - V\psi^*\psi + \frac{i\hbar}{2} (\psi^*\dot{\psi} - \psi\dot{\psi}^*).$$

Use the result obtained in the first part of the problem to show, that this Lagrangian density leads to the Schrödinger equation

$$-\frac{\hbar^2}{2m} \nabla^2\psi + V\psi = i\hbar \frac{\partial\psi}{\partial t}.$$

Calculate the canonical impulse densities corresponding to fields ψ and ψ^* (the calculation is reasonable, since the equations of motion, which one obtains by assuming ψ and ψ^* independent are valid).

2. Verify the conservation law (112)

$$\frac{\partial}{\partial t} \left[\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)} \frac{\partial u}{\partial x_j} \right] + \sum_i \frac{\partial}{\partial x_i} \left[\frac{\partial\mathcal{L}}{\partial(\partial u/\partial x_i)} \frac{\partial u}{\partial x_j} - \delta_{ij}\mathcal{L} \right] = -\frac{\partial\mathcal{L}}{\partial x_j}.$$

3. Study which of the three possible conservation laws discussed in the lectures are satisfied, if the equation of motion for the Lagrangian density is

- a) wave equation i.e. the Lagrangian density is

$$\mathcal{L} = \frac{\sigma}{2} \left(\frac{\partial u}{\partial t} \right)^2 - \frac{\tau}{2} \nabla u \cdot \nabla u.$$

- b) Schrödinger equation i.e. the Lagrangian density is

$$\mathcal{L} = -\frac{\hbar^2}{2m} \nabla\psi \cdot \nabla\psi^* - V\psi^*\psi + \frac{i\hbar}{2} (\psi^*\dot{\psi} - \psi\dot{\psi}^*).$$

Amuse yourself by considering the quantities \mathcal{H} , \mathcal{P}_j and π , and their physical meaning, when the corresponding conservation laws are satisfied.