1. The relativistic motion of a rocket is based on the conservation of four-momentum, when no external forces are directed to the rocket. Show that the conservation of four-momentum for the system, which includes the rocket and the outgoing gases leads to differential equation
$$m\frac{dv}{dm} + a(1 - \frac{v^2}{c^2}) = 0, \tag{1}$$
where $a \ll c$ is the velocity of the outgoing gases relative to the rocket, m is the mass of the rocket and v is the velocity of the rocket.
The differential equation (1) is separable and the integrations can be made using partial fraction decomposition. Let's assume that initially the rocket is at rest and has mass m_0. Show that the solution to differential equation (1) is
$$\beta = \frac{v}{c} = \frac{1 - (\frac{m}{m_0})^{\frac{2a}{c}}}{1 + (\frac{m}{m_0})^{\frac{2a}{c}}}.$$

2. Let's study an action
$$S = -\int (m_0 c + A) ds,$$
where $A = A(x^0, \vec{x})$. Calculate the canonical momentum, energy and equations of motion. The term
$$-\int A ds$$
in the action must correspond to some kind of potential, since the first term is the action for free particles. What do you have to assume to obtain familiar results from the analytical mechanics?

3. Let's consider two electromagnetic four-potentials A^i and A'^i with scalar and vector potentials: $\phi = -\vec{E}\cdot\vec{r}$, $\vec{A} = \frac{1}{2}\vec{B}\times\vec{r}$, $\phi' = \phi$ and $\vec{A}' = -By\vec{i}$. Let's assume that $\vec{E}$, $\vec{B}$ and B in the expressions above are constants. Show that electric and magnetic fields for both four-potentials are also constants.
Let's assume that $\vec{B} = B\vec{k}$ in the vector potential $\vec{A}$. Show that in this case the electric and magnetic fields are same for both four-potentials and the gauge transformation between these two four-potentials can be made with the function
$$f = -xy\frac{B}{2}.$$