

1. Let's consider a uniform stretched string initially at rest. The string is flicked at point $x = a$, after which the system will evolve according to the equations of motions derived from the Lagrangian. Let's assume, that the perturbation happens at time $t = 0$, and can be approximated as

$$\dot{u}(x, 0) = v_0 \delta(x - a).$$

Write down the function $u(x, t)$, which gives us the oscillation amplitudes at all eigenfrequencies.

2. Go through the derivation of the Lagrange equation given in lectures (equation (93))

$$\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial t)} + \sum_i \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial(\partial u / \partial x_i)} - \frac{\partial \mathcal{L}}{\partial u} = 0,$$

and show that the boundary condition (equation (92) in lectures)

$$\sum_i n_i \frac{\partial \mathcal{L}}{\partial(\partial u / \partial x_i)} \delta u = 0$$

should be required everywhere on the boundary of V to get the result. Note that the parameters n_i are the components of the unit vector \hat{n} perpendicular to the surface element at a given point of the boundary of V .

3. Let's consider the functional

$$F(u) = \frac{1}{2} \int dx \int dy (\nabla u \cdot \nabla u + \lambda u^2)$$

with a given boundary conditions. Find out the differential equation, which function $u(x, y)$ must satisfy to give a minimum for the functional $F(u)$.

Discretize the functional $F(u)$ using points $(x_i, y_j) = (a_i, a_j)$. Using the discrete version $F(u_{ij})$ (short-hand notation $F(u_{ij})$ is used to mean that F is a function of all independent variables u_{ij}), derive the equation, which the parameters u_{ij} must satisfy to give a minimum for function $F(u_{ij})$.

4. Consider the functional of the previous problem with $\lambda = 0$. Let's define a current \vec{j} to be

$$\vec{j} = \nabla u(x, y).$$

Show that

$$\nabla \cdot \vec{j} = 0,$$

if $u(x, y)$ satisfies the differential equation got in the previous problem. Transform the current to discrete lattice in the way, that the sum of all currents coming to a given lattice point is zero (assuming that u_{ij} satisfies the discrete equation got in the previous problem).