1. Consider a Lagrangian density, which is expressed in the form
$$\mathcal{L} = \mathcal{L}(\psi, \nabla\psi, \frac{\partial\psi}{\partial t}, \psi^*, \nabla\psi^*, \frac{\partial\psi^*}{\partial t}; x, t),$$
where $\psi = a + ib$ is a complex function, whose real and imaginary parts are independent field variables.
Show that the equations of motion for a and b are equivalent with the equations of motion, which one obtains by simply taking ψ and ψ^* as independent field variables.
Let's now consider the Lagrangian density
$$\mathcal{L} = -\frac{\hbar^2}{2m}\nabla\psi \cdot \nabla\psi^* - V\psi^*\psi + \frac{i\hbar}{2}(\psi^*\dot{\psi} - \psi\dot{\psi}^*).$$
Use the result obtained in the first part of the problem to show, that this Lagrangian density leads to the Schrödinger equation
$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}.$$
Calculate the canonical impulse densities corresponding to fields ψ and ψ^* (the calculation is reasonable, since the equations of motion, which one obtains by assuming ψ and ψ^* independent are valid).

2. Verify the conservation law (112)
$$\frac{\partial}{\partial t}\left[\frac{\partial\mathcal{L}}{\partial(\partial u/\partial t)}\frac{\partial u}{\partial x_j}\right] + \sum_i \frac{\partial}{\partial x_i}\left[\frac{\partial\mathcal{L}}{\partial(\partial u/\partial x_i)}\frac{\partial u}{\partial x_j} - \delta_{ij}\mathcal{L}\right] = -\frac{\partial\mathcal{L}}{\partial x_j}.$$

3. Study which of the three possible conservation laws discussed in the lectures are satisfied, if the equation of motion for the Lagrangian density is

   a) wave equation i.e. the Lagrangian density is
$$\mathcal{L} = \frac{\sigma}{2}\left(\frac{\partial u}{\partial t}\right)^2 - \frac{\tau}{2}\nabla u \cdot \nabla u.$$

   b) Scrödinger equation i.e. the Lagrangian density is
$$\mathcal{L} = -\frac{\hbar^2}{2m}\nabla\psi \cdot \nabla\psi^* - V\psi^*\psi + \frac{i\hbar}{2}(\psi^*\dot{\psi} - \psi\dot{\psi}^*).$$

   Amuse yourself by considering the quantities $\mathcal{H}$, $\mathcal{P}_j$ and π, and their physical meaning, when the corresponding conservation laws are satisfied.