1. The electron of a hydrogen atom is in the $4f$ state. To be more precise, its wave function is $\psi_{431}(r,\theta,\varphi) = R_{43}(r)Y_1^3(\theta,\varphi)$. Find the eigenvalues of the operators H, L^2, and L_z in this state.

2. Let us assume that a hydrogen atom is in the 3p state. Show that the radial part of its wave function is

$$\frac{1}{r}u_{31}(r) = \frac{4}{81\sqrt{6}}\, e^{-\frac{r}{3}}\, r(6-r).$$

3. The solutions of a diffential equation

$$\frac{d^2w}{dr^2} - 2\sqrt{-E}\,\frac{dw}{dr} + \left(\frac{2}{r} - \frac{\ell(\ell+1)}{r^2}\right) w = 0,$$

are polynomials

$$w(r) = \sum_{k=\ell+1}^{\infty} a_k r^k.$$

Show that the coefficients of the polynomial satisfy the recurrence relation

$$a_{k+1} = 2\frac{k\sqrt{-E}-1}{k(k+1)-\ell(\ell+1)}a_k.$$

4. Calculate the normalization factor N_{nl} for the Hydrogen atom.

$$I = N_{nl}^2 \int_0^\infty r^{2l+2} \exp^{-\frac{2r}{n}} \left[L_{n-l-1}^{2l+1}\left(\frac{2r}{n}\right)\right]^2 dr.$$

Use the recurrence relation

$$tL_k^\alpha = (2k+\alpha+1)L_k^\alpha - (k+1)L_{k+1}^\alpha - (k+a)L_{k-1}^\alpha.$$

5. Calculate the expectation value of potential energy for the hydrogen atom and express the result in natural units. Calculate also $\langle v^2\rangle$ using $\langle V\rangle + \langle T\rangle = E_n$. Finally, calculate the speed $v = \sqrt{\langle v^2\rangle}$ in SI units.

6. Determine the energy levels of a two-dimensional hydrogen atom. That is, determine the energy levels of a two-dimensional system comprising a proton and an electron. Note that the potential function of the system is

$$V(r) = -\frac{1}{4\pi\epsilon_0}\frac{e^2}{r},$$

where r is the distance between the proton and the electron.
Hint: See the PDF file "instructions for ex11p6". The file is located in the same directory as this exercise.