1. Show that the operators
   a) $p = -i\hbar\nabla$,
   b) x, the position operator,
   c) $H = -\frac{\hbar^2}{2m}\nabla^2 + V$, where V is a real function,
   are Hermitian.

2. Show that a linear combination of two square-integrable, complex-valued functions is also square-integrable.

3. a) Calculate the density $\rho(k_F)$ of the one-dimensional Fermi gas as a function of the Fermi wave vector k_F.
   b) Calculate the density of states $g(E)$ as a function of energy E in one and two dimensions.

4. Let us consider a two-state system. We denote the eigenstates of the Hamiltonian by $|\psi_i\rangle$ and the corresponding eigenvalues by $\hbar\omega_i$. That is,
$$H|\psi_i\rangle = \hbar\omega_i|\psi_i\rangle, \quad \text{where } i \in \{1,2\}. \tag{1}$$
   We denote the state of the system at time $t = t_0$ by
$$|\Psi(t_0)\rangle = c_1(t_0)|\psi_1\rangle + c_2(t_0)|\psi_1\rangle. \tag{2}$$
   We want to measure a dynamical quantity β to which corresponds a Hermitian operator B. We denote the eigenstates of B by $|u_j\rangle$ and the corresponding eigenvalues by b_j. That is,
$$B|u_j\rangle = b_j|u_j\rangle, \quad \text{where } j \in \{1,2\}. \tag{3}$$
   We assume that the eigenvalues of both H and B are nondegenerate (i.e. $\omega_1 \neq \omega_2$ and $b_1 \neq b_2$).

   a) Let us assume that B and H commute. What are the probabilities for obtaining b_1 and b_2 when we measure the dynamical quantity β at time $t \neq t_0$? Do these probabilities depend on time?
   Hint: It follows from our assumptions that B and H have the same eigenstates.

   b) Like a) but now B and H do not commute.
   Hint: For a discussion on the two-state system, see e.g. Wikipedia, two-state system.

5. a) Show that a wave packet $\psi(x,t)$, with a weight function $g(k)$ of Gaussian shape, centered around $k = k_0$, is at time $t = 0$ itself a Gaussian function, centered at $x = 0$:

$$\psi(x,0) = \left(\frac{2}{\pi a^2}\right)^{1/4} e^{ik_0 x} e^{-x^2/a^2}.$$

The constant a describes the spread of the Gaussian weight function. The rest of the constants are due to normalization.

b) A particle is described by a Gaussian wave packet. Is the particle then in a momentum eigenstate? Is it in an energy eigenstate of a free particle? Justify your answers.