Klein-Gordon's equation

We consider the scalar field $\phi(x)$ which, according to its definition, behaves under Lorentz transformation like

$$\phi'(x') = \phi(x).$$

Now

$$\mathcal{L} = \mathcal{L}(\phi, \partial\phi/\partial x_\mu).$$

Since we want

- the Lagrangian density to be invariant under Lorentz transformations
- a linear wave equation,

the Lagrangian density can contain only the terms

$$\phi^2 \text{ and } \frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu}.$$

One possible form for the Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^2\right).$$

Substituting this into the Euler-Lagrange equation

$$\frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0,$$

we get

$$-\frac{\partial}{\partial x_\mu}\left(\frac{\partial\phi}{\partial x_\mu}\right) + \mu^2\phi = 0.$$

If we employ the notation

$$\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

we end up with the *Klein-Gordon* equation

$$\Box\phi - \mu^2\phi = 0.$$

Heuristic derivation

We substitute into the relativistic energy-momentum relation

$$E^2 - |\boldsymbol{p}|^2c^2 = m^2c^4$$

the operators

$$E \mapsto i\hbar\frac{\partial}{\partial t}, \quad p_k \mapsto -i\hbar\frac{\partial}{\partial x_k},$$

and get

$$\left(-\frac{\partial^2}{c^2\partial t^2} + \nabla^2 - \frac{m^2c^2}{\hbar^2}\right)\phi = 0.$$

When we set

$$\mu = \frac{mc}{\hbar},\ [\mu] = \frac{1}{\text{length}},$$

we end up with the Klein-Gordon equation.

There are no sources in the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\phi}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^2\right)$$

so the solution describes a free field. We include the term

$$\mathcal{L}_{\text{int}} = -\phi\rho,$$

where ρ is the (usually position dependent) density of the source. The field equation is now

$$\Box\phi - \mu^2\phi = \rho.$$

When we choose

$$\rho = G\delta(\boldsymbol{x})$$

and seek for a stationary solution we end up with the equation

$$(\nabla^2 - \mu^2)\phi = G\delta(\boldsymbol{x}).$$

We substitute ϕ using its Fourier transform

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^{2/3}}\int d^3k e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\tilde{\phi}(\boldsymbol{k}),$$

where

$$\tilde{\phi}(\boldsymbol{k}) = \frac{1}{(2\pi)^{3/2}}\int d^3x\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\phi(\boldsymbol{x}).$$

We end up with the algebraic equation

$$(-\boldsymbol{k}^2 - \mu^2)\tilde{\phi}(\boldsymbol{k}) = \frac{G}{(2\pi)^{3/2}}$$

of the Fourier components. Its solution is

$$\tilde{\phi}(\boldsymbol{k}) = -\frac{G}{(2\pi)^{2/3}}\frac{1}{k^2+\mu^2}.$$

Taking the Fourier transform we get the solution

$$\phi(\boldsymbol{x}) = -\frac{G}{4\pi}\frac{e^{-\mu r}}{r},$$

known as the *Yukawa potential.* Let's suppose that the meson field of a nucleon at the point $\boldsymbol{x}_1$ satisfies the equations

$$(\nabla_2^2 - \mu^2)\phi = G\delta(\boldsymbol{x}_1 - \boldsymbol{x}_2).$$

Its solution is thus the Yukawa potential.

$$\phi(\boldsymbol{x}_2) = -\frac{G}{4\pi}\frac{e^{-\mu|\boldsymbol{x}_2-\boldsymbol{x}_1|}}{|\boldsymbol{x}_2-\boldsymbol{x}_1|}.$$

Because the Hamiltonian density was

$$\mathcal{H} = \dot{\eta}\pi - \mathcal{L},$$

the Hamiltonian density of the interaction is

$$\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}$$

and the total interaction Hamiltonian

$$H_{\text{int}} = \int \mathcal{H}_{\text{int}}\, d^3x = \int \phi\rho\, d^3x.$$

We see that the interaction energy of nucleons located at the points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is

$$H_{\text{int}}^{(1,2)} = -\frac{G}{4\pi}\frac{e^{-\mu|\boldsymbol{x}_2-\boldsymbol{x}_1|}}{|\boldsymbol{x}_2-\boldsymbol{x}_1|}.$$

Note Unlike in the Coulomb case, this interaction is atractive and short ranged.
In the reality there are 3 mesons (π^+, π^0, π^-), with different charges but with (almost) equal masses, consistent with the thory. We expand our theory so that we consider two real fields, ϕ_1 and ϕ_2, for two particles with equal masses. From these we construct the complex fields

$$\begin{aligned}\phi &= \frac{\phi_1 + i\phi_2}{\sqrt{2}} \\ \phi^* &= \frac{\phi_1 - i\phi_2}{\sqrt{2}}.\end{aligned}$$

The Lagrangian density for the free fields can be written using either the complex or real fields:

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\left(\frac{\partial\phi_1}{\partial x_\mu}\frac{\partial\phi_1}{\partial x_\mu} + \mu^2\phi_1^2\right) - \frac{1}{2}\left(\frac{\partial\phi_2}{\partial x_\mu}\frac{\partial\phi_2}{\partial x_\mu} + \mu^2\phi_2^2\right) \\ &= -\left(\frac{\partial\phi^*}{\partial x_\mu}\frac{\partial\phi}{\partial x_\mu} + \mu^2\phi^*\phi\right).\end{aligned}$$

Considering the fields ϕ and ϕ^* independent we get two Euler-Lagrange equations

$$\begin{aligned}\frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)} - \frac{\partial\mathcal{L}}{\partial\phi} &= 0 \\ \frac{\partial}{\partial x_\mu}\frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)} - \frac{\partial\mathcal{L}}{\partial\phi^*} &= 0,\end{aligned}$$

which can be further written as two Klein-Gordon equations

$$\begin{aligned}\Box\phi^* - \mu^2\phi^* &= 0 \\ \Box\phi - \mu^2\phi &= 0.\end{aligned}$$

We define the *first order gauge transformation* so that the fields transform under it like

$$\begin{aligned}\phi' &= e^{i\lambda}\phi \\ \phi^{*\prime} &= e^{-i\lambda}\phi^*,\end{aligned}$$

when λ is a real parameter. Let λ be now an arbitrary, infinitesimally small, number. Then

$$\begin{aligned}\delta\phi &= i\lambda\phi \\ \delta\phi^* &= -i\lambda\phi^*.\end{aligned}$$

The Lagrangian density transforms then as

$$\begin{aligned}\delta\mathcal{L} &= \left[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\delta\left(\frac{\partial\phi}{\partial x_\mu}\right)\right] \\ &\quad + \left[\frac{\partial\mathcal{L}}{\partial\phi^*}\delta\phi^* + \frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\delta\left(\frac{\partial\phi^*}{\partial x_\mu}\right)\right] \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\right)\right]\delta\phi \\ &\quad + \left[\frac{\partial\mathcal{L}}{\partial\phi^*} - \frac{\partial}{\partial x_\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\right)\right]\delta\phi^* \\ &\quad + \frac{\partial}{\partial x_\mu}\left[\frac{\partial\mathcal{L}}{\partial(\partial\phi/\partial x_\mu)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial\phi^*/\partial x_\mu)}\delta\phi^*\right] \\ &= -i\lambda\frac{\partial}{\partial x_\mu}\left(\frac{\partial\phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial\phi}{\partial x_\mu}\right).\end{aligned}$$

In a small neighborhood of the solutions ϕ and ϕ^* the Lagrangian density is invarinat so we must have

$$\delta\mathcal{L} = 0.$$

Thus we get

$$\frac{\partial s_\mu}{\partial x_\mu} = 0,$$

where

$$s_\mu = i\left(\frac{\partial\phi^*}{\partial x_\mu}\phi - \phi^*\frac{\partial\phi}{\partial x_\mu}\right).$$

We see that

- a complex field ϕ is associated with a conserved four-vector density s_μ,
- if we exchange $\phi \longleftrightarrow \phi^*$, then $s_\mu \longleftrightarrow -s_\mu$.

We interpret this so that

- s_μ is the charge current density,
- ϕ carries the charge e,
- ϕ^* carries the charge $-e$,
- the previous real field corresponds to neutral mesons.