"2ND" QUANTIZATION OF DIRAC FIELD

- OUR AIM IS TO CONSTRUCT FORMALISM IN WHICH e^- AND e^+ CAN BE CREATED AND DESTROYED FREELY (UNDER PAULI PRINCIPLE)
- WE PROCEED ANALOGOUSLY TO THE SPIN 1 CASE
- THE CLASSICAL LIMIT N_{e^-} ; $N_{e^+} \to \infty$ IS NOT OBVIOUS HERE, WE WILL WORRY ABOUT THAT LATER...

- FREE FIELD LAGRANGIAN

$$\mathcal{L} = -c\hbar\bar{\psi}\gamma_\mu\frac{\partial}{\partial x_\mu}\psi - mc^2\bar{\psi}\psi$$

$$= -c\hbar\bar{\psi}_\alpha(\gamma_\mu)_{\alpha\beta}\frac{\partial}{\partial x_\mu}\psi_\beta - mc^2\delta_{\alpha\beta}\bar{\psi}_\alpha\psi_\beta$$

IS LORENTZ INVARIANT SCALAR DENSITY

- EACH COMPONENT (OF ψ) IS INDEPENDENT FIELD VARIABLE THAT SATISFIES EULER LAGRANGE EQ.

$$\frac{\partial\mathcal{L}}{\partial\bar{\psi}_\alpha} = 0 \quad \Rightarrow \left(\gamma_\mu\frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi = 0$$

• THE EQUATION FOR $\bar{\Psi}$ IS OBTAINED WITH SUBSTITUTION

$$-c\hbar \bar{\Psi}_\alpha (\gamma_\nu)_{\alpha\beta} \frac{\partial}{\partial x_\nu} \Psi_\beta \rightarrow c\hbar \left(\frac{\partial}{\partial x_\nu} \bar{\Psi} \right)_\alpha (\gamma_\nu)_{\alpha\beta} \Psi_\beta$$

• CANONICAL MOMENTUM

$$\Pi_\beta = \frac{\partial \mathcal{L}}{\partial (\partial \Psi_\beta / \partial t)} = i\hbar \bar{\Psi}_\alpha (\gamma_4)_{\alpha\beta}$$

$$= i\hbar \Psi^\dagger_\beta$$

• HAMILTONIAN DENSITY

$$\mathcal{H} = c\Pi_\beta \frac{\partial \Psi_\beta}{\partial x_0} - \mathcal{L}$$

$$= c\hbar \left(i\Psi^\dagger \frac{\partial \Psi}{\partial x_0} - i\bar{\Psi}\gamma_4 \frac{\partial \Psi}{\partial x_0} + \bar{\Psi}\gamma_k \frac{\partial}{\partial x_k} \Psi + mc^2 \bar{\Psi}\Psi \right)$$

$$\mathcal{H} = \Psi^\dagger\left(-i\hbar c\bar{\alpha}\cdot\bar{\nabla} + \beta mc^2\right)\Psi$$

AND HAMILTONIAN

$$H = \int \Psi^\dagger\left(-i\hbar c\bar{\alpha}\cdot\bar{\nabla} + \beta mc^2\right)\Psi\, d^3x \qquad (*)$$

CLASS.

- AGAIN, WE EXPAND THE SOLUTION IN TERMS OF FREE PARTICLE PLANE WAVES (WHICH FORM COMPLETE ORTHONORMAL BASIS)

$$\Psi(\bar{x},t)_{\text{op}} = \frac{1}{\sqrt{V}}\sum_{\bar{p}}\sum_{r=1}^{4}\sqrt{\frac{mc^2}{|E|}}\, b_{\bar{p}}^{(r)}(t)\, u^{(r)}(\bar{p})\, e^{i\bar{p}\cdot x/\hbar}$$

- Ψ_{opq} IS AN OPERATOR ACTING ON STATE KETS

$b, b^{\dagger}$ ARE ANNIHILATION/CREATION OPERATORS DEFINED BEFORE

- SINGLE ELECTRON WITH $\bar{p}, r$: $|1_{\bar{p},r}\rangle = b_{\bar{p}}^{(r)\dagger}(0)|0\rangle$

$t=0$

- LET US ASSUME THAT THE HAMILTONIAN OPERATOR ENTAILS THE CLASSICAL FORM (*)

$$H_{opq} = \frac{1}{V}\int \sum_{\bar{p}} \sum_{\bar{p}'} \sum_{r=1}^{4} \sum_{r'=1}^{4} \left(\sqrt{\frac{mc^2}{|E|}}\, b_{\bar{p}}^{(r)\dagger} u^{(r)\dagger}(\bar{p}) e^{-i\bar{p}\cdot\bar{x}/\hbar} \right)$$

$$\times \left(-i\hbar c \bar{\alpha}\cdot\bar{\nabla} + \beta mc^2 \right) \left(\sqrt{\frac{mc^2}{|E'|}}\, b_{\bar{p}'}^{(r')} u^{(r')}(\bar{p}') e^{i\bar{p}'\cdot\bar{x}/\hbar} \right) d^3x$$

$$= \sum_{\bar{p}} \sum_{\bar{p}'} \sum_{r} \sum_{r'} \delta_{\bar{p}\bar{p}'} \frac{mc^2 E'}{\sqrt{|EE'|}}\, b_{\bar{p}}^{(r)\dagger} b_{\bar{p}'}^{(r')} u^{(r)\dagger}(\bar{p}) u^{(r')}(\bar{p}')$$

$$= \sum_{\bar{p}} \sum_{r=1,2} |E|\, b_{\bar{p}}^{(r)\dagger} b_{\bar{p}}^{(r)} - \sum_{\bar{p}} \sum_{r=3,4} |E|\, b_{\bar{p}}^{(r)\dagger} b_{\bar{p}}^{(r)}$$

SINCE $u^{(r)\dagger}(\bar{p})\, u^{(r')}(\bar{p}) = 0$ IF $r = r'$

(CORRELATIONS)

$$u^{(r)\dagger}(\bar{p})\, u^{(r)}(\bar{p}) = \frac{|E|}{mc^2}$$

$$E' = \pm\sqrt{|\bar{p}'|^2 c^2 + m^2 c^4} \quad \begin{cases} r' = 1,2 \\ r' = 3,4 \end{cases}$$

H_{opq} OPERATES ON OCCUPATION SPACE STATE VECTORS

- LET US WORK OUT THE TIME-DEPENDENCE OF $b^\dagger(t)$ $b(t)$

- HEISENBERG:

$$\frac{d}{dt} b_{\bar{p}}^{(r)} = \dot{b}_{\bar{p}}^{(r)} = \frac{i}{\hbar}\left[H, b_{\bar{p}}^{(r)}\right] = \mp \frac{i}{\hbar} b_{\bar{p}}^{(r)} |E|$$

$-$: $r = 1,2$
$+$: $r = 3,4$

$$\dot{b}_{\bar{p}}^{(r)\dagger} = \frac{i}{\hbar}\left[H, b_{\bar{p}}^{(r)\dagger}\right] = \pm \frac{i}{\hbar} b_{\bar{p}}^{(r)} |E|$$

$+$: $r = 1,2$
$-$: $r = 3,4$

(COMES WITH APPLYING

$$[AB, C] = A\{B, C\} - \{A, C\} B \;)$$

<u>THUS:</u>

$$b_{\bar{p}}^{(r)}(t) = b_{\bar{p}}^{(r)}(0) e^{\mp i |E| t/\hbar} \quad \begin{cases} r = 1,2 & : -1 \\ r = 3,4 & : + \end{cases}$$

$\mp \to \pm$ FOR $b_{\bar{p}}^{(r)\dagger}$

$$\psi(\bar{x},t)_{opq} = \frac{1}{V}\sum_{\bar{p}}\sqrt{\frac{mc^2}{|E|}}$$

$$\times\left(\sum_{r=1,2} b_{\bar{p}}^{(r)}(0)\,u^{(r)}(\bar{p})\,e^{i\bar{p}\cdot\bar{x}/\hbar - i|E|t/\hbar}\right.$$

$$\left.+\sum_{r=3,4} b_{\bar{p}}^{(r)}(0)\,u^{(r)}(\bar{p})\,e^{i\bar{p}\cdot\bar{x}/\hbar + i|E|t/\hbar}\right)$$

- THIS SATISFIES THE DIRAC EQUATION

- NOTICE, UNLIKE $a, a^\dagger$, $b, b^\dagger$ ARE NOT LINEAR COMBINATIONS OF P, Q WITH $[P,Q] = i\hbar$

LET US RE-DEFINE

$b_{\bar{p}}^{(r)}(0) \Longrightarrow b_{\bar{p}}^{(r)}$

$b_{\bar{p}}^{(r)\dagger}(0) \Longrightarrow b_{\bar{p}}^{(r)}$

SINCE THE TIME EVOLUTION IS EXPLICT FROM NOW ON

$$Q = e\int \psi^\dagger \psi \, d^3x$$

$$= e\sum_{\bar{p}}\sum_{\bar{p}'}\sum_{r}\sum_{r'}\left(\frac{mc^2}{\sqrt{|EE'|}}\right)\delta_{\bar{p}\bar{p}'}\, b_{\bar{p}}^{(r)}\, b_{\bar{p}'}^{(r')}\, u^{(r)\dagger}(\bar{p})\, u^{(r')}(\bar{p}')$$

$$= e\sum_{\bar{p}}\sum_{r} b_{p}^{(r)\dagger} b_{p}^{(r)}$$

$$\bar{p}_{TOT} = \sum_{\bar{p}}\sum_{r}\bar{p}\, b_{\bar{p}}^{(r)}\, b_{\bar{p}}^{(r)}$$

- NOTICE, WE STILL HAVE THE E<0 SOLUTIONS ALIVE IN THE SUMMATIONS

LET US GET RID OF THEM...

LET US DEFINE CREATION AND ANNIHILATION OPERATORS FOR POSITRONS SO THAT

$$d_{\bar{p}}^{(1)+} = -b_{-\bar{p}}^{(4)} \qquad b_{\bar{p}}^{(1)} = b_{\bar{p}}^{(r=1)}$$

$$d_{\bar{p}}^{(2)+} = +b_{-\bar{p}}^{(3)} \qquad b_{\bar{p}}^{(2)} = b_{\bar{p}}^{(r=2)}$$

FOR e^+ FOR e^-

$$v^{(1)}(\bar{p}) = -u^{(4)}(-\bar{p}) \qquad u^{(1)}(\bar{p}) = u^{(r=1)}(\bar{p})$$

$$v^{(2)}(\bar{p}) = u^{(3)}(-\bar{p}) \qquad u^{(2)}(\bar{p}) = u^{(r=2)}(\bar{p})$$

I.E. WE HAVE 2-COMPONENT OPERATORS

$$b^{(s)}, b^{(s)+}, d^{(s)}, d^{(s)+} \quad ; s = 1,2$$

LET US VERBALIZE:

ANNIHILATION OF

$e(E<0, p=-\bar{p}, s=\downarrow)$

$\equiv$ CREATION OF

$e^{+}(E>0, p=+\bar{p}, s=\uparrow)$

QUITE OBVIOUSLY

$$\{d_{\bar{p}}^{(s)}, d_{\bar{p}'}^{(s')+}\} = \delta_{ss'}\delta_{\bar{p}\bar{p}'}$$

$$\{d_{\bar{p}}^{(s)}, d_{\bar{p}'}^{(s')}\} = \{d_{\bar{p}}^{(s)+} d_{\bar{p}'}^{(s')+}\} = 0$$

AND ANY

$\{\tilde{b}, \tilde{d}\} = 0$ $\boxed{\tilde{b} = b, b^{+} \quad \tilde{d} = d, d^{+} \text{ WITH ANY } s, \bar{p}}$

WE HAVE

$$(i\gamma^\nu p_\nu + mc)u^{(s)}(\bar{p}) = 0$$

$$(-i\gamma^\nu p_\nu + mc)v^{(s)}(\bar{p}) = 0$$

AND THE ON RELATIONS

$$u^{(s')+}(\bar{p})u^{(s)}(\bar{p}) = \delta_{ss'}\frac{E}{mc^2}$$

$$v^{(s')+}(\bar{p})v^{(s)}(\bar{p}) = \delta_{ss'}\frac{E}{mc^2}$$

$$v^{(s')+}(-\bar{p})u^{(s)}(\bar{p}) = u^{(s')+}(-\bar{p})v^{(s)}(\bar{p}) = 0$$

ABOVE AND FROM NOW ON

<u>E IS ALWAYS POSITIVE</u>

$$\psi_{opq}(\bar{x},t)=\frac{1}{\sqrt{V}}\sum_{\bar{p}}\sum_{s=1,2}\left(\sqrt{\frac{mc^2}{E}}\, b^{(s)}_{\bar{p}}\, u^{(s)}(\bar{p})\, e^{i(\bar{p}\cdot\bar{x}-Et)/\hbar}\right.$$

$$\left.+\, d^{(s)}_{\bar{p}}\, v^{(s)}(\bar{p})\, e^{-i(\bar{p}\cdot\bar{x}-Et/\hbar)}\right)$$

$$\bar{\psi}_{opq}(\bar{x},t)=\frac{1}{\sqrt{V}}\sum_{\bar{p}}\sum_{s=1,2}\left(\sqrt{\frac{mc^2}{E}}\left(d^{(s)}_{\bar{p}}\, \bar{v}^{(s)}(\bar{p})\, e^{i(\bar{p}\cdot\bar{x}-Et)/\hbar}\right.\right.$$

$$\left.+\, b^{(s)}_{\bar{p}}\, \bar{u}^{(s)}_{q}(\bar{p})\, e^{-i(\bar{p}\cdot\bar{x}-Et)/\hbar}\right)$$

$E=\sqrt{|\bar{p}|^2c^2+m^2c^4}$ ALWAYS

$$H=\sum_{\bar{p}}\sum_{s}E\left(b^{(s)\dagger}_{\bar{p}}b^{(s)}_{\bar{p}}-d^{(s)}_{-\bar{p}}d^{(s)\dagger}_{-\bar{p}}\right)$$

$$=\sum_{\bar{p}}\sum_{s}E\left(b^{(s)\dagger}_{\bar{p}}b^{(s)}_{\bar{p}}+d^{(s)\dagger}_{\bar{p}}d^{(s)}_{\bar{p}}-1\right)$$

$$Q=\sum_{\bar{p}}\sum_{s}e\left(b^{(s)\dagger}_{\bar{p}}b^{(s)}_{\bar{p}}+d^{(s)}_{-\bar{p}}d^{(s)\dagger}_{-\bar{p}}\right)$$

$$=e\sum_{\bar{p}}\sum_{s}\left(b^{(s)\dagger}_{\bar{p}}b^{(s)}_{\bar{p}}-d^{(s)\dagger}_{\bar{p}}d^{(s)}_{\bar{p}}+1\right)$$

WHERE

$$\boxed{N_{\bar{p}}^{(e^-,s)} = b_{\bar{p}}^{(s)\dagger} b_{\bar{p}}^{(s)} \quad \Big| \quad N_{\bar{p}}^{(e^+,s)} = d_{\bar{p}}^{(s)\dagger} d_{\bar{p}}^{(s)}}$$

- LOOKING AT THE HAMILTONIAN, WE SEE THAT VACUUM IS THE GROUND STATE (LOWEST E) WITH $E=-\infty$
- BUT WE CAN CHOOSE THE GROUND STATE ENERGY OF OUR SYSTEM AND SO LET US REDEFINE

$$H = \sum_{\bar{p}} \sum_{s} E\left(N_{\bar{p}}^{e^-,s} + N_{\bar{p}}^{e^+,s}\right)$$

$$\rightarrow \quad H|0\rangle = 0$$

• SIMILARLY FOR CHARGE

$$Q = -|e| \sum_{\bar{p}} \sum_{s} \left(N_{\bar{p}}^{e^-,s} - N_{\bar{p}}^{e^+,s} \right)$$

TOTAL MOMENTUM BECOMES

$$\bar{P}_{TOT} = \sum_{\bar{p}} \sum_{r} \bar{p}\, b_{\bar{p}}^{(r)\dagger} b_{\bar{p}}^{(r)}$$

$$= \sum_{\bar{p}} \sum_{s} \bar{p} \left(b_{\bar{p}}^{(s)\dagger} b_{\bar{p}}^{(s)} + d_{-\bar{p}}^{(s)} d_{-\bar{p}}^{(s)\dagger} \right)$$

$$= \sum_{\bar{p}} \sum_{s} \bar{p} \left[b_{\bar{p}}^{(s)\dagger} b_{\bar{p}}^{(s)} + (-\bar{p}) \left(-d_{\bar{p}}^{(s)\dagger} d_{\bar{p}}^{(s)} + 1 \right) \right]$$

$$= \sum_{\bar{p}} \sum_{s} \bar{p} \left(N_{\bar{p}}^{e^-,s} + N_{\bar{p}}^{e^+,s} \right)$$

$$\sum_{\bar{p}} \bar{p} = 0$$

· CONNECTION BETWEEN
C AND q NUMBER SPACES

C NUMBER SPACE: Ψ = WAVE FUNCTION
q NUMBER SPACE: Ψ = OPERATOR

FOR A SINGLE PARTICLE STATE

$$\Psi_c = \langle 0 | \Psi_q | b_{\bar{p}}^{(r)\dagger} \Phi_0 \rangle$$

$$= \frac{1}{\sqrt{V}} \langle 0 | \sum_{\bar{p}'} \sum_{r'} \sqrt{\frac{mc^2}{|E|}}\, b_{\bar{p}'}^{(r')} u^{(r')}(\bar{p}') e^{ip^{N'}x_N} b_{\bar{p}}^{(r)\dagger} | 0 \rangle$$

$$= \sqrt{\frac{mc^2}{|E|V}} \sum_{\bar{p}'} \sum_{r'} u^{(r')}(\bar{p}') e^{ip^{N'}x_N} \underbrace{\langle 0 | b_{\bar{p}'}^{(r')} b^{(r)\dagger} | 0 \rangle}_{= \delta_{\bar{p},\bar{p}'} \delta_{r,r'}}$$

$$= \sqrt{\frac{mc^2}{|E|V}}\, u^{(r)}(\bar{p}) e^{ip^N x_N}$$

$$= \Psi_c$$