1. Warm up
   a) Show that the eigenvalues of a Hermitian operator A are real and that the eigenkets of A corresponding to different eigenvalues are orthogonal.
   b) Show that if the state ket $|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$ is normalized then the expansion coefficients $c_{a'}$ must satisfy $\sum_{a'} |c_{a'}|^2 = 1$.

Solution:

a)

i) A number c is shown to be real if $c^\star = c$. Let us study Hermitian operator A to whom holds $A^\dagger = A$ and an eigenstate $|a\rangle$ corresponding an eigenvalue a, such that $A|a\rangle = a|a\rangle$. Based on evalution of the inner product

$$c = \langle a |A| a\rangle = a\langle a|a\rangle$$

the eigenvalue a has the expression

$$a = \frac{c}{\langle a|a\rangle}.$$

Let us now study what is $a^\star$?

$$a^\star = \frac{c^\star}{(\langle a|a\rangle)^\star} = \frac{(\langle a |A| a\rangle)^\star}{\langle a|a\rangle} = \frac{(\langle a |A^\dagger| a\rangle)}{\langle a|a\rangle} = \frac{\langle a |A| a\rangle}{\langle a|a\rangle} = a.$$

Now it has been proven that a Hermitian operator A has real eigenvalues.

ii) To prove that the eigenkets of A corresponding to different eigenvalues are orthogonal (i.e. $\langle b|a\rangle = 0$), we examine the inner product $\langle b |A| a\rangle$ between two eigenstates $|a\rangle$ and $|b\rangle$ corresponding different eigenvalues a and b, $(a \neq b)$. The inner product can be evaluated two different ways:

$$d = \langle b |A| a\rangle = a\langle b|a\rangle$$

$$d = \langle b |A| a\rangle = \left\langle b \left|A^\dagger\right| a\right\rangle = b\langle b|a\rangle.$$

In the latter, the hermicity of A is applied. Now the above two expression are subtracted from each other

$$0 = (a - b)\langle b|a\rangle$$

which implies in case of $a \neq b$ that $\langle b|a\rangle = 0$.

b) First of all,

$$\langle\alpha|\alpha\rangle = \left(\sum_{a'} c^\star_{a'} \langle a'|\right)\left(\sum_{a''} c_{a''} |a''\rangle\right) = \sum_{a',a''} c^\star_{a'} c_{a''} \langle a'|a''\rangle = \sum_{a'} |c_{a'}|^2 \in \mathbb{R},$$

then the normalization condition

$$|\langle\alpha|\alpha\rangle|^2 = (\langle\alpha|\alpha\rangle)^2 = 1$$

straight implies that

$$1 = \langle\alpha|\alpha\rangle = \sum_{a'} |c_{a'}|^2.$$

The previous proof about orthogonality (a.ii) holds also for a degenerate case, then corresponding an eigenvalue, say, b, we have a set of eigenstates $|b_1\rangle, |b_2\rangle, \ldots, |b_j\rangle$, but anyway all of them are orthogonal to some other eigenstate $|a\rangle$ corresponding eigenvalue $a \neq b$.

2. Prove the Theorem 1 from lecture notes:
If both of the basis $\{|a'\rangle\}$ and $\{|b'\rangle\}$ are orthonormalized and complete then there exists a unitary operator U so that

$$|b_1\rangle = U|a_1\rangle, \quad |b_2\rangle = U|a_2\rangle, \quad |b_3\rangle = U|a_3\rangle, \quad \ldots \tag{1}$$

(Unitary operator: $U^\dagger U = UU^\dagger = 1$)
Solution:
The proof has three stages: construction of operator U, proof of property (1) and proof of unitarity. Construction procedure is rather easy, we would like to build an operator that projects the basis state $|a_j\rangle$ to basis state $|b_j\rangle$:

$$U = \sum_j |b_j\rangle\langle a_j|.$$

To show that the property (1) holds, operator U operates on an arbitrary basis state $|a_k\rangle$:

$$U|a_k\rangle = \sum_j |b_j\rangle\langle a_j|a_k\rangle = \sum_j |b_j\rangle \delta_{jk} = |b_k\rangle$$

where the orthonormality of basis $\{|a'\rangle\}$ plays a role. The unitarity is checked via brute calculation:

$$\begin{aligned} UU^\dagger = \sum_j^\infty |b_j\rangle\langle a_j| \sum_i^\infty (|b_i\rangle\langle a_i|)^\dagger &= \sum_j^\infty |b_j\rangle\langle a_j| \sum_i^\infty |a_i\rangle\langle b_i| \\ &= \sum_{ij} = |b_j\rangle \underbrace{\langle a_j|a_i\rangle}_{\delta_{ij}} \langle b_i| = \underbrace{\sum_j |b_j\rangle\langle b_j|}_{\text{completeness of } \{|b'\rangle\}} = 1. \end{aligned}$$

3. Consider the spin operators S_x, S_y and S_z in the $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis
   a) Write out the operators S_x, S_y and S_z in the $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis.
   b) Compute the commutators $[S_x, S_y]$ and $[S^2, S_x]$ as well as anticommutator $\{S_x, S_y\}$.
   c) Let us define the ladder operators $S_\pm = S_x \pm \mathrm{i} S_y$. Compute $S_\pm |S_z;\uparrow\rangle$ and $S_\pm |S_z;\downarrow\rangle$.

Solution:

Let us first summarize the $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis represented with the help eigenstates of S_x and S_y operators with proper phase choice (see. e.g. J. J. Sakurai, Modern Quantum Mechanics, p. 28):

$$\begin{aligned} |S_z;\uparrow\rangle &= \frac{1}{\sqrt{2}}|S_x;\uparrow\rangle + \frac{1}{\sqrt{2}}|S_x;\downarrow\rangle \qquad & |S_z;\uparrow\rangle &= \frac{1}{\sqrt{2}}|S_y;\uparrow\rangle + \frac{1}{\sqrt{2}}|S_y;\downarrow\rangle \\ |S_z;\downarrow\rangle &= \frac{1}{\sqrt{2}}|S_x;\uparrow\rangle - \frac{1}{\sqrt{2}}|S_x;\downarrow\rangle \qquad & |S_z;\downarrow\rangle &= -\frac{\mathrm{i}}{\sqrt{2}}|S_y;\uparrow\rangle + \frac{\mathrm{i}}{\sqrt{2}}|S_y;\downarrow\rangle \end{aligned} \tag{2}$$

Representation of an operator B in the $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis

$$B = \sum_{j=\uparrow\downarrow}\sum_{k=\uparrow\downarrow} |S_z;j\rangle\langle S_z;j|B|S_z;k\rangle\langle S_z;k| = \sum_{j=\uparrow\downarrow}\sum_{k=\uparrow\downarrow} B_{jk}|S_z;j\rangle\langle S_z;k|$$

and in matrix representation we use the following convention with the indecies

$$B_{jk} = \langle S_z;j|B|S_z;k\rangle$$

$$B = \begin{pmatrix} B_{\uparrow\uparrow} & B_{\uparrow\downarrow} \\ B_{\downarrow\uparrow} & B_{\downarrow\downarrow} \end{pmatrix} = \begin{pmatrix} \langle S_z;\uparrow|B|S_z;\uparrow\rangle & \langle S_z;\uparrow|B|S_z;\downarrow\rangle \\ \langle S_z;\downarrow|B|S_z;\uparrow\rangle & \langle S_z;\downarrow|B|S_z;\downarrow\rangle \end{pmatrix}.$$

a) Basis representation for operator S_z is after above definitions just the calculation of matrix elements B_{jk}:

$$\begin{aligned} S_z &= \begin{pmatrix} \langle S_z;\uparrow|S_z|S_z;\uparrow\rangle & \langle S_z;\uparrow|S_z|S_z;\downarrow\rangle \\ \langle S_z;\downarrow|S_z|S_z;\uparrow\rangle & \langle S_z;\downarrow|S_z|S_z;\downarrow\rangle \end{pmatrix} \\ &= \frac{\hbar}{2}\begin{pmatrix} \langle S_z;\uparrow|S_z;\uparrow\rangle & -\langle S_z;\uparrow|S_z;\downarrow\rangle \\ \langle S_z;\downarrow|S_z;\uparrow\rangle & -\langle S_z;\downarrow|S_z;\downarrow\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned}$$

To do the same for $S_{x,y}$ we resort to relations (2) and find out that

$$\begin{aligned} S_x|S_z;\uparrow\rangle &= \frac{\hbar}{2}|S_z;\downarrow\rangle \qquad & S_y|S_z;\uparrow\rangle &= \mathrm{i}\frac{\hbar}{2}|S_z;\downarrow\rangle \\ S_x|S_z;\downarrow\rangle &= \frac{\hbar}{2}|S_z;\uparrow\rangle \qquad & S_y|S_z;\downarrow\rangle &= -\mathrm{i}\frac{\hbar}{2}|S_z;\uparrow\rangle \end{aligned}$$

which shows that

$$S_x = \begin{pmatrix} \langle S_z;\uparrow|S_x|S_z;\uparrow\rangle & \langle S_z;\uparrow|S_x|S_z;\downarrow\rangle \\ \langle S_z;\downarrow|S_x|S_z;\uparrow\rangle & \langle S_z;\downarrow|S_x|S_z;\downarrow\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \begin{pmatrix} \langle S_z;\uparrow|S_y|S_z;\uparrow\rangle & \langle S_z;\uparrow|S_y|S_z;\downarrow\rangle \\ \langle S_z;\downarrow|S_y|S_z;\uparrow\rangle & \langle S_z;\downarrow|S_y|S_z;\downarrow\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}.$$

b) As we now have the representations of operators S_i in the S_z eigenstate basis we can use them to calculate the (anti)commutators.

$$
\begin{aligned}
[S_x, S_y] &= S_x S_y - S_y S_x = \frac{\hbar^2}{4}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} - \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) \\
&= \mathrm{i}\hbar S_z
\end{aligned}
$$

(The general rule goes $[S_i, S_j] = \mathrm{i}\hbar\epsilon_{ijk}S_k$, where ϵ_{ijk} is the Levi-Civita permutation symbol.)

Then it happens out that $S_i^2 = \hbar^2/4$ for all $i = x, y, z$, therefore $S^2 = S_x^2 + S_y^2 + S_z^2 = 3\hbar^2/4$ and it is clear that $[S^2, S_x] = 3\hbar^2[I, S_x]/4 = 0$. When calculating $[S_x, S_y]$ one notices that $S_x S_y = -S_y S_x$ which implies that $\{S_x, S_y\} = 0$.

c) Since matricies are handy objects, let us express ladder opertors $S_\pm$ also in the familiar $\{|S_z;\uparrow\rangle, |S_z;\downarrow\rangle\}$ basis: $S_\pm = S_x \pm \mathrm{i}S_y$.

$$
\begin{aligned}
S_+ &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
S_- &= \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

or

$$
S_+ = \hbar\,|S_z;\uparrow\rangle\langle S_z;\downarrow| \qquad\qquad S_- = \hbar\,|S_z;\downarrow\rangle\langle S_z;\uparrow|.
$$

and operations to S_z eigenstates result

$$
\begin{aligned}
S_+|S_z;\uparrow\rangle &= 0 & S_-|S_z;\uparrow\rangle &= \hbar\,|S_z;\downarrow\rangle \\
S_+|S_z;\downarrow\rangle &= \hbar\,|S_z;\uparrow\rangle & S_-|S_z;\downarrow\rangle &= 0
\end{aligned}
$$

Now, the physical meaning of the ladder operators can be read. Operator S_+ raises the spin component by $\hbar$ and if the spin component cannot be raised further, we get null state. Similarly, S_- lowers the spin component by $\hbar$. Both these operators are non-Hermitian.

4. Prove the Theorem 2 from lecture notes:
If T is a unitary matrix, then the matrices X and $T^\dagger XT$ have the same trace and the same eigenvalues.
Solution:
   i) Trace of a matrix X is the sum of its diagonal elements: $\mathrm{Tr}\,(X) = \sum_i X_{ii}$ and as an reminder the matrix multiplication expressed in index notation goes $(AB)_{ij} = \sum_k A_{ik} B_{kj}$. The unitarity of T has then index expression:

$$TT^\dagger = 1 \quad \Rightarrow \quad \sum_k T_{ik} T^\dagger_{kj} = \delta_{ij}$$

$$T^\dagger T = 1 \quad \Rightarrow \quad \sum_k T^\dagger_{ik} T_{kj} = \delta_{ij}$$

   With these in our mind we are ready to prove the trace invariance:

$$\begin{aligned} \mathrm{Tr}\left(T^\dagger XT\right) = \sum_i (T^\dagger XT)_{ii} &= \sum_i \sum_j \sum_k T^\dagger_{ij} X_{jk} T_{ki} \\ &= \sum_k \sum_j X_{jk} \underbrace{\sum_i T_{ki} T^\dagger_{ij}}_{\delta_{kj}} \\ &= \sum_k X_{kk} = \mathrm{Tr}\,(X)\,. \end{aligned}$$

   ii) The matrix X has eigenvalues $\{a_1, a_2, \ldots, a_n\}$ and corresponding eigenvectors $\{|a_1\rangle, |a_2\rangle, \ldots, |a_n\rangle\}$. By constructing a new set of vectors such that $|b_j\rangle = T^\dagger |a_j\rangle$ and evaluating

$$T^\dagger XT\,|b_j\rangle = T^\dagger XTT^\dagger\,|a_j\rangle = T^\dagger X\,|a_j\rangle = a_j T^\dagger\,|a_j\rangle = a_j\,|b_j\rangle\,,$$

   we observe that $|b_j\rangle$ are the eigenvectors of matrix $T^\dagger XT$ corresponding the same eigenvalues $\{a_1, a_2, \ldots, a_n\}$. Thus matricies X and $T^\dagger XT$ has the same eigenvalues if T is a unitary matrix.

5. The translation operator for a finite (spatial) displacement is given by

$$\mathcal{T}(\mathbf{l}) = \exp\left(-\frac{\mathrm{i}}{\hbar}\mathbf{p}\cdot\mathbf{l}\right)$$

where $\mathbf{p}$ is the momentum operator and $\mathbf{l}$ the displacement vector.

a) Evaluate $[x_i, \mathcal{T}(\mathbf{l})]$.

b) How does the expectation value $\langle x\rangle$ of the position operator change under the translation?

Solution:

a) As introduced in lectures the effect of a finite spatial displacement by $\mathbf{l}$ is $\mathcal{T}(\mathbf{l})\,|\mathbf{x}\rangle = |\mathbf{x}+\mathbf{l}\rangle$. When evaluating commutator one should remember when x_i is an operator and when a pure number, to distinguish these two cases I use now notation $\hat{x}_i$ for operator and x_i for number. Let us introduce arbitrary state $|\alpha\rangle$ having presentation in configuration space $|\alpha\rangle = \int \mathrm{d}\mathbf{x}\;|\mathbf{x}\rangle\,\langle\mathbf{x}|\alpha\rangle$

$$\begin{aligned}
[\hat{x}_i, \mathcal{T}(\mathbf{l})]\,|\alpha\rangle &= \hat{x}_i\mathcal{T}(\mathbf{l})\,|\alpha\rangle - \mathcal{T}(\mathbf{l})\hat{x}_i\,|\alpha\rangle \\
&= \hat{x}_i \int \mathrm{d}\mathbf{x}\;|\mathbf{x}+\mathbf{l}\rangle\,\langle\mathbf{x}|\alpha\rangle - \mathcal{T}(\mathbf{l}) \int \mathrm{d}\mathbf{x}\,x_i\,|\mathbf{x}\rangle\,\langle\mathbf{x}|\alpha\rangle \\
&= \int \mathrm{d}\mathbf{x}\,(x_i + l_i)\,|\mathbf{x}+\mathbf{l}\rangle\,\langle\mathbf{x}|\alpha\rangle - \int \mathrm{d}\mathbf{x}\,x_i\,|\mathbf{x}+\mathbf{l}\rangle\,\langle\mathbf{x}|\alpha\rangle \\
&= \int \mathrm{d}\mathbf{x}\,l_i\,|\mathbf{x}+\mathbf{l}\rangle\,\langle\mathbf{x}|\alpha\rangle = l_i\mathcal{T}(\mathbf{l})\,|\alpha\rangle\,.
\end{aligned}$$

Now we say that $[\hat{x}_i, \mathcal{T}(\mathbf{l})] = l_i\mathcal{T}(\mathbf{l})$, which is an operator identity since it holds for an arbitrary state. Furthermore, we know by generalising the result that $\hat{\mathbf{x}}\mathcal{T}(\mathbf{l}) - \mathcal{T}(\mathbf{l})\hat{\mathbf{x}} = \mathbf{l}\mathcal{T}(\mathbf{l})$ or $\hat{\mathbf{x}}\mathcal{T}(\mathbf{l}) = \mathcal{T}(\mathbf{l})(\mathbf{l}+\hat{\mathbf{x}})$.

b) Before the translation, the expectation value of position operator $\hat{\mathbf{x}}$ for an arbitrary state is $\langle x\rangle = \langle\alpha\,|\hat{\mathbf{x}}|\,\alpha\rangle$ and after the translation

$$\begin{aligned}
\langle\alpha_{\mathbf{l}}\,|\hat{\mathbf{x}}|\,\alpha_{\mathbf{l}}\rangle = \left\langle\alpha\left|\mathcal{T}^\dagger(\mathbf{l})\hat{\mathbf{x}}\mathcal{T}(\mathbf{l})\right|\alpha\right\rangle &= \left\langle\alpha\left|\mathcal{T}^\dagger(\mathbf{l})\mathcal{T}(\mathbf{l})(\mathbf{l}+\hat{\mathbf{x}})\right|\alpha\right\rangle \\
&= \langle\alpha\,|\mathbf{l}+\hat{\mathbf{x}}|\,\alpha\rangle \\
&= \mathbf{l} + \langle x\rangle,
\end{aligned}$$

which is not any big surprise.